UPPSALA UNIVERSITET
Matematiska institutionen
M. Klimek

Prov i matematik
Kurs: 1MA022
2016-06-02

Complex Analysis

Writing time: 14:00–19:00.
Other than writing utensils and paper, no help materials are allowed.

---

1. Suppose that

$$u(x, y) = x^2 - y^2 + 2x + 1 + \log(x^2 + y^2), \qquad (x, y) \neq (0, 0).$$

Show that u is harmonic. Let $D = \mathbb{C} \setminus (-\infty, 0]$. Find an analytic function $f : D \longrightarrow \mathbb{C}$ such that $f(1) = 4$ and $\mathrm{Re} f(z) = u(x, y)$ for $z = x + iy \in D$. Write a formula for f as a function of z.

2. Find a conformal mapping that transforms the domain

$$\{z \in \mathbb{C} : \mathrm{Im}\, z > 0\} \cup \{z \in \mathbb{C} : |z| < 1\}$$

onto the infinite horizontal strip $\{z \in \mathbb{C} : -1 < \mathrm{Im}\, z < 1\}$.

Hint: If Q is a quadrant of the plane, describe the set $\{\mathrm{Log}\, z \,:\, z \in Q\}$, where Log is the principal branch of the complex logarithm.

3. Find the Laurent series expansion of the function

$$f(z) = \frac{(z - i)^3 - (z + i)^3}{(z^2 + 1)^3}$$

in the domain $D = \{z \in \mathbb{C} \,:\, |z| > 1\}$.

4. Use the residue theorem to calculate

$$\int_0^\infty \frac{x - \sin x}{x^3(x^2+1)} dx.$$

Hint: Consider the complex function

$$f(z) = \frac{z + i(e^{iz} - 1)}{z^3(z^2+1)}.$$

Show that this function has a simple pole at $z = 0$.

5. Let $\gamma : [a,b] \longrightarrow \mathbb{D}$ be a piecewise smooth curve parameterizing the boundary of a bounded domain $D \subset \mathbb{C}$. Assume that f is a complex function which is analytic in a neighbourhood of the closure of D and such that $f(z) \neq 0$ at all $z \in \partial D$. Consider the curve $\Gamma(t) = f(\gamma(t))$, $t \in [a,b]$. Prove that the number of zeros of f in D (counted according to their multiplicities) is given by the winding number $W(\Gamma, 0)$.

6. Let m, n be natural numbers and let $\alpha \geq 1$ be a constant. Consider the function

$$g(z) = \sum_{k=0}^{m} \frac{z^k}{k!} - e^\alpha z^n, \qquad z \in \mathbb{C}.$$

Show that this function has n zeros in the unit disc, irrespective of the choice of the numbers m and α.

7. Find a formula for an analytic function $f : \mathbb{C} \setminus \{0, i, -i\} \longrightarrow \mathbb{C}$ which has the following properties:

- f has zeros of order 3 at ± 2;
- f has double poles at $\pm i$;
- f has a pole of order 3 at 0 with residue 1;
- f has a simple zero at infinity.

Is there more than one function with these properties? Justify your answer.

8. Suppose that $f : \mathbb{C} \longrightarrow \mathbb{C}$ is an analytic function such that for some constant $M > 0$ and for all $z \in \mathbb{C}$ the following inequality is satisfied:

$$|f(z)| \leq M + \log(1 + |z|).$$

Show that then f must be a constant function. Use this conclusion to show that there are no non-constant harmonic functions $u : \mathbb{C} \longrightarrow \mathbb{R}$ satisfying the inequality

$$e^{u(z)} \leq M + \log(1 + |z|), \qquad z \in \mathbb{C}.$$

GOOD LUCK!