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Prov i matematik
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Complex Analysis

Writing time: 14:00–19:00.
Other than writing utensils and paper, no help materials are allowed.

1. Suppose that

$$u(x,y) = x^2 - y^2 + 2x + 1 + \log(x^2 + y^2), \qquad (x,y) \neq (0,0).$$

Show that u is harmonic. Let $D = \mathbb{C} \setminus (-\infty, 0]$. Find an analytic function $f : D \longrightarrow \mathbb{C}$ such that $f(1) = 4$ and $\mathrm{Re} f(z) = u(x,y)$ for $z = x + iy \in D$. Write a formula for f as a function of z.

2. Find a conformal mapping that transforms the domain

$$\{z \in \mathbb{C} : \mathrm{Im}\, z > 0\} \cup \{z \in \mathbb{C} : \, |z| < 1\}$$

onto the infinite horizontal strip $\{z \in \mathbb{C} : -1 < \mathrm{Im}\, z < 1\}$.

Hint: If Q is a quadrant of the plane, describe the set $\{\mathrm{Log}\, z \,:\, z \in Q\}$, where Log is the principal branch of the complex logarithm.

3. Find the Laurent series expansion of the function

$$f(z) = \frac{(z-i)^3 - (z+i)^3}{(z^2+1)^3}$$

in the domain $D = \{z \in \mathbb{C} \,:\, |z| > 1\}$.

4. Use the residue theorem to calculate

$$\int_0^\infty \frac{x - \sin x}{x^3(x^2+1)} dx.$$

Hint: Consider the complex function

$$f(z) = \frac{z + i(e^{iz} - 1)}{z^3(z^2+1)}.$$

Show that this function has a simple pole at $z = 0$.

5. Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a piecewise smooth curve parameterizing the boundary of a bounded domain $D \subset \mathbb{C}$. Assume that f is a complex function which is analytic in a neighbourhood of the closure of D and such that $f(z) \neq 0$ at all $z \in \partial D$. Consider the curve $\Gamma(t) = f(\gamma(t))$, $t \in [a, b]$. Prove that the number of zeros of f in D (counted according to their multiplicities) is given by the winding number $W(\Gamma, 0)$.

6. Let m, n be natural numbers and let $\alpha \geq 1$ be a constant. Consider the function

$$g(z) = \sum_{k=0}^{m} \frac{z^k}{k!} - e^\alpha z^n, \qquad z \in \mathbb{C}.$$

Show that this function has n zeros in the unit disc, irrespective of the choice of the numbers m and α.

7. Find a formula for an analytic function $f : \mathbb{C} \setminus \{0, i, -i\} \longrightarrow \mathbb{C}$ which has the following properties:

- f has zeros of order 3 at ± 2;
- f has double poles at $\pm i$;
- f has a pole of order 3 at 0 with residue 1;
- f has a simple zero at infinity.

Is there more than one function with these properties? Justify your answer.

8. Suppose that $f : \mathbb{C} \longrightarrow \mathbb{C}$ is an analytic function such that for some constant $M > 0$ and for all $z \in \mathbb{C}$ the following inequality is satisfied:

$$|f(z)| \leq M + \log(1 + |z|).$$

Show that then f must be a constant function. Use this conclusion to show that there are no non-constant harmonic functions $u : \mathbb{C} \longrightarrow \mathbb{R}$ satisfying the inequality

$$e^{u(z)} \leq M + \log(1 + |z|), \qquad z \in \mathbb{C}.$$

GOOD LUCK!

SOLUTIONS

1. Since $\log(x^2+y^2) = \mathrm{Re}(2\mathrm{Log}\, z)$ and $\mathrm{Log}\, 1 = 0$ it is enough to find the harmonic conjugate of $\tilde{u}(x,y) = x^2 - y^2 + 2x + 1$. Obviously $\Delta \tilde{u} = 2 - 2 = 0$. According to the Cauchy-Riemann equations $\tilde{u}_x = 2x+2 = \tilde{v}_y$ and $\tilde{u}_y = -2y = -\tilde{v}_x$. The last one implies that $\tilde{v} = 2xy + \phi(y)$ for some real-valued function ϕ. Thus $\tilde{v}_y = 2x + \phi'(y) = 2x + 2$, and hence $\phi(y) = 2y + \text{const}$. So $\tilde{v}(x,y) = 2xy + 2y$ as it is supposed to vanish at $1 + i0$. Finally

$$x^2 - y^2 + 2x + 1 + i(2xy + 2y) = (x+iy)^2 + 2(x+iy) + 1 = (z+1)^2,$$

and so the answer is $f(z) = (z+1)^2 + 2\mathrm{Log}\, z$.

2. Let $Q_I, Q_{II}, Q_{III}, Q_{IV}$ denote the 1st, 2nd, 3rd and 4th quadrant in the plane. We want to map $Q_I \cup Q_{II} \cup D(0,1)$ onto $Q_{II} \cup Q_{III}$. The composition of the following mappings will do:

- $z \mapsto z + 1$ maps $Q_I \cup Q_{II} \cup D(0,1)$ onto $Q_I \cup Q_{II} \cup D(1,1)$;
- $z \mapsto 1/z$ maps $Q_I \cup Q_{II} \cup D(1,1)$ onto $Q_{III} \cup Q_{IV} \cup \{z \in \mathbb{C} : \mathrm{Re}\, z > 1/2\}$;
- $z \mapsto z - 1/2$ maps $Q_{III} \cup Q_{IV} \cup \{z \in \mathbb{C} : \mathrm{Re}\, z > 1/2\}$ onto $Q_{III} \cup Q_{IV} \cup Q_I$;
- $z \mapsto \mathrm{Log}\, z$ maps $Q_{III} \cup Q_{IV} \cup Q_I$ onto the infinite strip $\{z \in \mathbb{C} : -\pi < \mathrm{Im}\, z < \pi/2\}$
- $z \mapsto 4(z + i\pi/4)/(3\pi)$ maps $\{z \in \mathbb{C} : -\pi < \mathrm{Im}\, z < \pi/2\}$ onto $\{z \in \mathbb{C} : -1 < \mathrm{Im}\, z < 1\}$.

The outcome is

$$f(z) = \frac{4}{3\pi}\left\{\mathrm{Log}\left[\frac{1}{2}\left(\frac{1-z}{1+z}\right)\right] + \frac{i\pi}{4}\right\}.$$

3. Clearly

$$f(z) = \frac{1}{(z+i)^3} - \frac{1}{(z-i)^3}, \qquad |z| > 1,$$

and

$$\left(\frac{1}{z \pm i}\right)'' = \frac{2}{(z \pm i)^3}.$$

If $|z| > 1$, then

$$\frac{1}{z \pm i} = \frac{1}{z} \cdot \frac{1}{1 + \left(\mp \frac{i}{z}\right)} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\mp \frac{i}{z}\right)^n = \sum_{m=1}^{\infty} (\mp i)^{m-1} z^{-m}.$$

Consequently

$$f(z) = \sum_{k=3}^{\infty} \frac{(k-2)(k-1)}{2} \left(i^{k-3} - (-i)^{k-3}\right) z^{-k}.$$

Note that

$$i^{k-3} - (-i)^{k-3} = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ 2i, & \text{if } k \text{ is even and is divisible by 4}, \\ -2i & \text{if } k \text{ is even and is not divisible by 4}. \end{cases}$$

4. Let $0 < r < 1 < R$. Note that the real function we are integrating is even, and hence the integrals over $[-R, -r]$ and $[r, R]$ are the same. Let

$$f(z) = \frac{z + i(e^{iz} - 1)}{z^3(z^2 + 1)}.$$

Note that

$$\frac{x - \sin x}{x^3(x^2 + 1)} = \operatorname{Re} f(x), \qquad x \in \mathbb{R}.$$

Apart form simple poles at $\pm i$, the function $f(z)$ has a simple pole at 0, because the numerator has a double zero at 0. If γ_r is the upper semicircle with center at 0, radius r, and clockwise orientation, then by the fractional residue theorem

$$\lim_{r \to 0} \int_{\gamma_r} f(z)\, dz = -\frac{\pi}{2}.$$

If Γ_R is the upper semicircle with center at 0, radius R, and counter-clockwise orientation, then

$$\lim_{R \to \infty} \int_{\Gamma_R} f(z)\, dz = 0.$$

By the ordinary residue theorem

$$\int_{[-R,-r]} f(z)\, dz + \int_{\gamma_r} f(z)\, dz + \int_{[r,R]} f(z)\, dz + \int_{\Gamma_R} f(z)\, dz = 2\pi i \operatorname{Res}[f, i] = -\frac{\pi}{e}.$$

By letting $r \to 0$ and $R \to \infty$, and then comparing the real parts, we get the answer as $\frac{\pi}{4} - \frac{\pi}{2e}$.

5. Let $\gamma : [a,b] \to \mathbb{C}$. We have

$$\begin{aligned} W(\Gamma,0) &= \frac{1}{2\pi i}\int_\Gamma \frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int_a^b \frac{\Gamma'(t)dt}{\Gamma(t)} = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\gamma'(t)dt}{f(\gamma(t))} \\ &= \int_\gamma \frac{f'(z)}{f(z)}dz = Z_f(D) \end{aligned}$$

according to the Argument Principle.

6. We use Rouche's Theorem with $f(z) = -e^\alpha z^n$ and $h(z) = g(z) - f(z)$. Then for z with modulus 1, we have

$$|h(z)| \le \sum_{k=0}^m \frac{1}{k!} < e \le e^\alpha = |f(z)|.$$

7. If the only zeros of f are at ± 2 and at ∞, then the function

$$h(z) := \frac{(z^2+1)^2 z^3}{(z^2-4)^3} f(z)$$

has only removable singularities and no zeros in $\mathbb{C}$. At ∞ it has a non-zero limit and so by Liouville's theorem it is a constant $c \neq 0$. Hence

$$f(z) = \frac{c(z^2-4)^3}{(z^2+1)^2 z^3}.$$

Since

$$1 = \mathrm{Res}[f,0] = \frac{1}{2}\left(\frac{c(z^2-4)^3}{(z^2+1)^2}\right)''\Big|_{z=0} = 176c,$$

it follows that $c = 1/176$.

8. If $R > 0$ and $n \in \mathbb{N}$, then by the given inequality and Cauchy's Estimates we have

$$\frac{|f^{(n)}(0)|}{n!} \le \frac{M + \log(1+R)}{R^n} \to 0 \text{ as } R \to \infty.$$

Thus the power series expansion of f about 0 reduces to a constant term. If v is a harmonic conjugate of u, then the second part follows from the first one applied to $f = \exp(u + iv)$.