UPPSALA UNIVERSITET
Matematiska institutionen
Jörgen Östensson

Prov i matematik
KandMa2 m.fl.
Complex Analysis
2017–05–31

Writing time: 08.00 – 13.00. Allowed aids: Writing materials. Each problem has a maximum credit of 5 points. Bonus points from the homework assignments will be added to your exam result. For the grades 3, 4 and 5 respectively, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be clearly written and properly explained.

1. Solve the equation

$$(2-i)\sin z+\cos z=2-i.$$

The answer should be given in the form $a+ib$, where a and b are real numbers.

2. Find all functions $f=u+iv$ which are analytic in $\mathbb{C}$ and have real part of the form

$$u(x,y)=x\,\phi(y),$$

where ϕ is a real-valued function of one variable of class C^2. The answer should be given as an expression in the variable $z=x+iy$.

3. Find a Möbius transformation which maps the region $\{z:|z|<2 \text{ and } |z-i|>1\}$ onto the region $\{w:0<\operatorname{Im}w<\pi\}$, and which fixes the point 0.

4. Compute the values of the following integrals:

(a) $\displaystyle\int_\Gamma \frac{1}{z+i}\,dz$, where Γ is the half-circle in the lower half-plane from $-\sqrt{3}$ to $\sqrt{3}$.

(b) $\displaystyle\int_C \frac{z^2}{(z+1)(z-1)^2}\,dz$, where C is the counterclockwise oriented circle $|z|=2$.

5. Find the Laurent series expansion of the function

$$f(z)=\frac{1}{z^2(z-1)}$$

in the annulus $1<|z+1|<2$.

6. Evaluate the integral

$$I=\int_{-\infty}^{\infty}\frac{\cos x}{e^x+e^{-x}}\,dx$$

by integrating the function $f(z)=\dfrac{e^{iz}}{e^z+e^{-z}}$ around the rectangle with vertices at $\pm R$ and $\pm R+i\pi$.

Turn page!

7. Determine the number of zeros of the polynomial $p(z) = z^6 + 17z^3 + 3z^2 + 2$ in the open square $\Omega = \{z = x + iy : |x| < 1 \text{ and } |y| < 1\}$.

8. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ denote an entire function satisfying the estimate

$$|f(z)| \leq M e^{|z|} \quad \text{for all } z \in \mathbb{C}$$

for some constant M. Prove that the coefficients a_n satisfy

$$|a_n| \leq M \left(\frac{e}{n}\right)^n, \quad n = 1, 2, 3, ...$$

GOOD LUCK!

Svar till tentamen i Complex Analysis 2017–05–31

1. $z = \dfrac{\pi}{4} + 2n\pi - i\dfrac{\ln 2}{2}$ and $z = \dfrac{\pi}{2} + 2n\pi, \quad n \in \mathbb{Z}.$

2. $f(z) = iAz^2 + Bz + iC$, where A, B, C are real constants.

3. $T(z) = \dfrac{2\pi i z}{z - 2i}.$

4. (a) $\dfrac{4\pi i}{3}$, (b) $2\pi i$.

5. $f(z) = -\displaystyle\sum_{n=1}^{\infty} \frac{n}{(z+1)^n} - \sum_{n=0}^{\infty} \frac{(z+1)^n}{2^{n+1}}.$

6. $I = \dfrac{\pi}{e^{\pi/2} + e^{-\pi/2}}.$

7. 3.