UPPSALA UNIVERSITET
Matematiska institutionen
Jörgen Östensson

Prov i matematik
KandMa2 m.fl.
Complex Analysis
2017–08–18

Writing time: 08.00 – 13.00. Allowed aids: Writing materials. Each problem has a maximum credit of 5 points. For the grades 3, 4 and 5 respectively, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be clearly written and properly explained.

1. Solve the equation

$$\cos z = i.$$

The answer should be given in the form $a + ib$, where a and b are real numbers.

2. Find all functions $f = u + iv$ which are analytic in $\mathbb{C}$ and satisfy

$$u + v = x^2 - y^2.$$

The answer should be given as an expression in the variable $z = x + iy$.

3. Find a Möbius transformation which maps the circle $|z-2+i| = \sqrt{5}$ onto the circle $|w+2| = 2$, and which maps the points 0 and $1 - i$ to the points 0 and -2, respectively.

4. The function

$$f(z) = \frac{\frac{1}{z^2} + \frac{1}{\pi^2}}{e^z - e^{-z}}$$

has a pole at the origin.

(a) Find the principal part of f at the origin, i.e. that part of the Laurent series of f in a punctured neighborhood of the origin which contains the negative powers of z.

(b) The Laurent series of f considered in (a) has region of convergence of the form $0 < |z| < R$. Determine R.

5. Let $a > 1$. Calculate the integral

$$\int_0^{2\pi} \frac{1}{(a + \cos\theta)^2}\, d\theta.$$

6. Determine the number of zeros of the polynomial

$$p(z) = z^5 + 10z^3 + 4z^2 + 9z + 1$$

in the first quadrant $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$.

Turn page!

7. Evaluate the integral

$$\int_0^\infty \frac{1-\cos x}{x^2(x^2+1)}\,dx.$$

8. Find all entire functions f such that

$$|f(z)| \geq \frac{1}{1+|z|^{2017}} \quad \text{for all } z \in \mathbb{C}.$$

GOOD LUCK!

Svar till tentamen i Complex Analysis 2017–08–18

1. $z = \pm\left(\dfrac{\pi}{2} - i\ln(\sqrt{2}+1)\right) + 2\pi n$, $n \in \mathbb{Z}$.

2. $f(z) = \dfrac{1}{2}(1+i)z^2 + (1-i)A$, where A is a real constant.

3. $T(z) = -\dfrac{4(1+i)z}{z+3+i}$.

4. (a) $\dfrac{1}{2z^3} + \left(\dfrac{1}{2\pi^2} - \dfrac{1}{12}\right)\dfrac{1}{z}$. (b) $R = 2\pi$.

5. $\dfrac{2\pi a}{(a^2-1)^{3/2}}$.

6. 1.

7. $\dfrac{\pi}{2e}$.

8. f has to be constant (a constant c with $|c| \geq 1$).