To Lösbold

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Prov i matematik
KandFy3, KandMa2 m.fl.
Complex Analysis
2018–05–31

Writing time: 08.00 – 13.00. Allowed aids: Writing materials. Each problem has a maximum credit of 5 points. Bonus points from the homework assignments will be added to your exam result. For the grades 3, 4 and 5 respectively, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be clearly written and properly explained.

1. Find all functions $f = u + iv$ which are analytic in $\mathbb{C}$ such that

$$u(x, y) = x^3 - 3xy^2 - 4xy + 3y.$$

The answer should be given as an expression in the variable $z = x + iy$.

2. Find a conformal mapping which maps the sector $\{z : \frac{\pi}{3} < \arg z < \frac{2\pi}{3}\}$ onto the open unit disk centered at the origin, so that the point i is mapped to the origin.

3. Find the Laurent series expansion of the function

$$f(z) = \frac{5z}{z^2 + z - 6}$$

in the annulus $1 < |z - 1| < 4$.

4. Calculate the value of the integral

$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{(x^2+1)^2}\,dx$$

for $\omega \in \mathbb{R}$.

5. Determine the number of zeros of the polynomial

$$p(z) = z^4 - z^3 - 2z^2 - 4z - 3$$

in the right half-plane $\operatorname{Re} z > 0$.

6. The function f is analytic in the whole complex plane with the exception of a double pole at the point $z = 1$, where the residue is 3. Furthermore, $f(0) = f'(0) = 1$ and

$$\lim_{z\to\infty} \frac{f(z)}{z} = 0.$$

Show that the function f is uniquely determined, and find f.

Turn page!

7. Let C_N denote the positively oriented boundary of the square whose edges lie along the lines

$$x = \pm(N + \tfrac{1}{2})\pi \quad \text{and} \quad y = \pm(N + \tfrac{1}{2})\pi,$$

where N is a positive integer.

(a) Show that

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left(\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2 \pi^2} \right).$$

(b) Carefully explain how it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

[Hint for (b): It holds that $|\sin z|^2 = \sin^2 x + \sinh^2 y$, $z = x + iy$.]

8. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function analytic in the disk $|z| < 1$ satisfying

$$|f(z)| \leq \frac{1}{1 - |z|}, \qquad |z| < 1.$$

Prove that the coefficients a_n satisfy the inequality

$$|a_n| \leq \frac{(n+1)^{n+1}}{n^n}, \quad n = 1, 2, 3, \ldots.$$

GOOD LUCK!

Tb Lösning

Svar till tentamen i Complex Analysis 2018–05–31

1. $f(z) = z^3 + 2iz^2 - 3iz + iC$, where C is a real constant.

2. $f(z) = \dfrac{z^3 + i}{z^3 - i}$.

3. $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} a_n (z-1)^n$, where $a_n = \begin{cases} \dfrac{3(-1)^n}{4^{n+1}} & , n \geq 0, \\ 2 & , n \leq -1. \end{cases}$

4. $\dfrac{\pi}{2}(1 + |\omega|)e^{-|\omega|}$.

5. 1.

6. $f(z) = \dfrac{2z^2 - z + 1}{(z-1)^2}$.

Exam 2018-05-31

① $u = x^3 - 3xy^2 - 4xy + 3y$

By the CR-eqns

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

This gives

$$\begin{cases} v_y = 3x^2 - 3y^2 - 4y & (1) \\ v_x = 6xy + 4x - 3 & (2) \end{cases}$$

$(1) \iff v = 3x^2y - y^3 - 2y^2 + \varphi(x)$

Insert into (2):

$$\cancel{6xy} + \varphi'(x) = \cancel{6xy} + 4x - 3$$

$\iff \varphi(x) = 2x^2 - 3x + C \; , \; C \in \mathbb{R}$

Thus, $f = x^3 - 3xy^2 - 4xy + 3y$
$+ i\left(3x^2y - y^3 - 2y^2 + 2x^2 - 3x + C\right)$

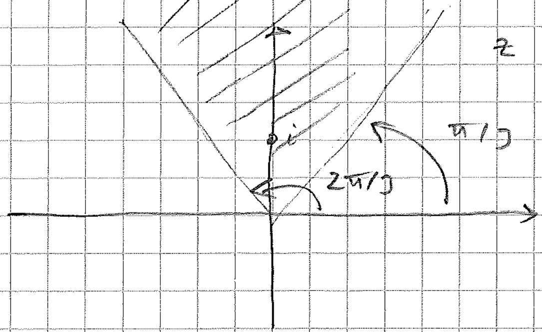
Note that

$$f(x + i0) = x^3 + i2x^2 - i3x + iC$$

By the uniqueness principle

$$f(z) = z^3 + i2z^2 - i3z + iC \; , \; C \in \mathbb{R}$$

② See figure:



Put $h(z) = z^3$. Clearly h maps the given sector conformally onto the lower half-plane $\operatorname{Im} w < 0$, and the point i to $-i$.

Now, let g be the Möbius mapping

$$g(w) = \frac{w+i}{w-i}$$

It sends the point $-i$ to 0 and the point i to ∞.

Since $-i$ & i are symmetric w.r.t. the real line, their images will be symmetric w.r.t. the image of the real line. It is therefore clear that g maps the real line onto a circle with center at the origin

Since $g(0) = -1$, the circle must

have radius 1.

I.e., g maps the real line onto

the unit circle, and the lower

half-plane onto the open unit disk

Thus, $f = g \circ h$ does the job.

Clearly,

$$f(z) = \frac{z^3 + i}{z^3 - i}$$

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(3) $$f(z) = \frac{5z}{z^2+z-6}$$

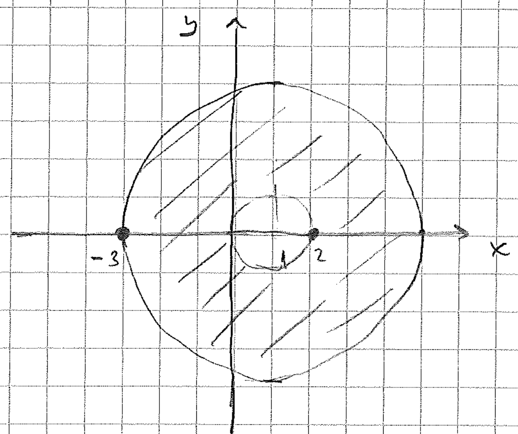
f is analytic except where

$$z^2+z-6=0 \iff (z+3)(z-2)=0$$

I.e. except at $z=2$ and $z=-3$

Therefore f is analytic in $1<|z-1|<4$.

See figure:



Next, note that

$$f(z) = \frac{A}{z+3} + \frac{B}{z-2}$$

where $$A = \left.\frac{5z}{z-2}\right|_{z=-3} = +3$$

$$B = \left.\frac{5z}{z+3}\right|_{z=2} = 2$$

I.e.

$$f(z) = \frac{2}{z-2} + \frac{3}{z+3}$$

Thus,

$$f(z) = \frac{2}{z-1-1} + \frac{3}{z-1+4} =$$

$$= \frac{1}{z-1} \cdot \frac{2}{1-\frac{1}{z-1}} + \frac{3}{4} \frac{1}{1+\frac{z-1}{4}} =$$

$$= \frac{1}{z-1} \cdot 2 \sum_{n=0}^{\infty} \frac{1}{(z-1)^n} + \frac{3}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{2}{(z-1)^{n+1}} + \sum_{n=0}^{\infty} \frac{3(-1)^n}{4^{n+1}} (z-1)^n$$

$$= \sum_{n=-\infty}^{\infty} a_n (z-1)^n ,$$

where

$$a_n = \begin{cases} \frac{3(-1)^n}{4^{n+1}} , & n \geq 0 \\ 2 , & n \leq -1 \end{cases}$$

(4) $$I = \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{(x^2+1)^2}\, dx$$

First note that $I = \int_{-\infty}^{\infty} \frac{\cos\omega x + i \sin\omega x}{(x^2+1)^2}\, dx$ odd

$$= \int_{-\infty}^{\infty} \frac{\cos\omega x}{(x^2+1)^2}\, dx; \text{ even in } \omega.$$

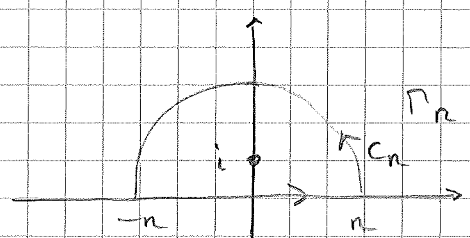
So suffices to consider $\omega \geq 0$.

Let $f(z) = \frac{e^{i\omega z}}{(z^2+1)^2}$.

Clearly f is analytic in $\mathbb{C} \setminus \{\pm i\}$;

f has double poles at $z = \pm i$.

Consider $\int_{\Gamma_R} f(z)\, dz$ with Γ_R as below:



By the residue thm

$$\int_{\Gamma_R} f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz =$$

$$= 2\pi i \cdot \mathrm{Res}(f, i)$$

We have that

$$\operatorname{Res}(f, i) = \lim_{z \to i} \frac{d}{dz}\left((z-i)^2 f(z)\right) =$$

$$= \frac{d}{dz} \frac{e^{i\omega z}}{(z+i)^2} \Bigg|_{z=i} =$$

$$= i\omega e^{i\omega z} \frac{1}{(z+i)^2} - 2 e^{i\omega z} \frac{1}{(z+i)^3} \Bigg|_{z=i} =$$

$$= i\omega e^{-\omega} \frac{1}{-4} - 2 e^{-\omega} \frac{1}{-8i} =$$

$$= -\frac{i}{4} (\omega + 1) e^{-\omega}$$

This holds for any $\omega (\geq 0)$, i.e. there is no need to consider $\omega = 0$ separately.

Thus,

$$\int_{\Gamma_R} f(z)\, dz = 2\pi i \cdot \left(-\frac{i}{4}\right)(\omega+1) e^{-\omega}$$

$$= \frac{\pi}{2} (1+\omega) e^{-\omega}$$

Now, by the ML-ineq., for $\omega \geq 0$

$$\left| \int_{C_R} f(z)\, dz \right| \leq \frac{1}{(R^2-1)^2} \cdot \pi R \to 0, \quad R \to \infty$$

(Also follows from Jordan's lemma)

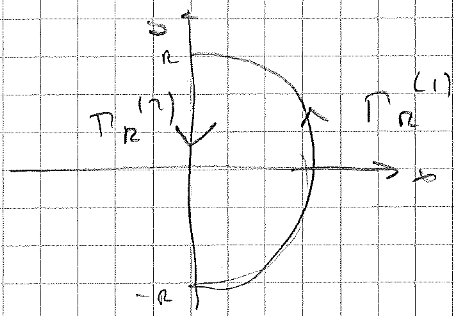
By evenness in ω, then letting $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{(x^2+1)^2}\, dx = \frac{\pi}{2} (1+|\omega|) e^{-|\omega|}.$$

(5)

$$p(z) = z^4 - z^3 - 2z^2 - 4z - 3$$

Apply the argument principle to the contour Γ_R below:



Clearly $\Delta_{\Gamma_R^{(1)}} \arg p \simeq 4\pi$ for large R

To investigate $\Delta_{\Gamma_R^{(2)}} \arg p$, note that

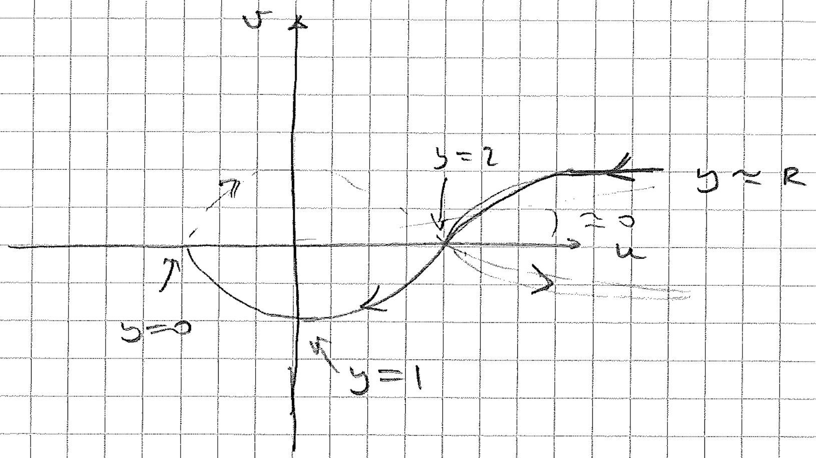
$$\begin{aligned} p(iy) &= y^4 + iy^3 + 2y^2 - i4y - 3 \\ &= y^4 + 2y^2 - 3 + i(y^3 - 4y) \\ &= (y^2-1)(y^2+3) + iy(y^2-4) \\ &= u(y) + iv(y) \end{aligned}$$

where

$$u(y) = (y^2-1)(y^2+3) = (y-1)(y+1)(y^2+3)$$

$$v(y) = y(y^2-4) = (y-2)y(y+2)$$

See figure:



Clearly, $\Delta_{\Gamma_R^{(1)}} \arg p \simeq -2\pi$

$\Rightarrow \Delta_{\Gamma_R} \arg p = 4\pi - 2\pi = 2\pi$

for large R.

By the argument principle, it follows that p has $\underline{1}$ zero in the right half-plane Re $z > 0$

($\Delta_{\Gamma_R} \arg p = 2\pi \cdot \#\{$zeros inside $\Gamma_R\}$ by analyticity of p)

(6)

The principal part of f at $z=1$ has the form

$$\frac{A}{(z-1)^2} + \frac{3}{z-1}$$

Thus, $g(z) := f(z) - \frac{A}{(z-1)^2} + \frac{3}{z-1}$

can be considered analytic in all of $\mathbb{C}$.

Now,

$$\lim_{z\to\infty} \frac{g(z)}{z} = \lim_{z\to\infty} \frac{f(z)}{z} = 0$$

$\rightarrow |g(z)| \leq |z|$ for $|z|$ large.

By (the extension) of Liouville's thm, it follows that g must be a polynomial of degree ≤ 1, i.e.

$g(z) = Dz + C$

But since $\lim_{z\to\infty} \frac{g(z)}{z} = 0$,

clearly $D=0$; i.e. $g(z) = C$

Hence, f must have the form

$$f(z) = \frac{A}{(z-1)^2} + \frac{3}{z-1} + C$$

It follows from $f(0) = f'(0) = 1$ that

$$\begin{cases} A - 3 + C = 1 \\ 2A - 3 = 1 \end{cases}$$

I.e. $A = 2$ and $C = 2$

Thus,

$$f(z) = \frac{2}{(z-1)^2} + \frac{3}{z-1} + 2 =$$

$$= \frac{2 + 3(z-1) + 2(z-1)^2}{(z-1)^2} =$$

$$= \frac{2 + 3z - 3 + 2(z^2 - 2z + 1)}{(z-1)^2} =$$

$$= \frac{2z^2 - z + 1}{(z-1)^2}$$

Hence, f is uniquely determined and given by

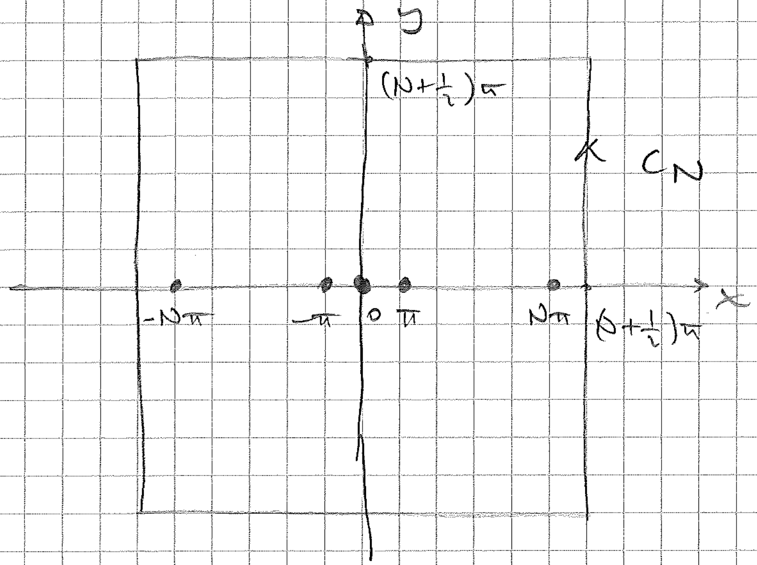
$$f(z) = \frac{2z^2 - z + 1}{(z-1)^2}.$$

(7) (c) Put $f(z) = \dfrac{1}{z^2 \sin z}$

f has a tripple pole at $z=0$,

and simple poles at $z = n\pi$, $n \in \mathbb{Z} \setminus \{0\}$.

See figure:



Let $n \in \mathbb{Z} \setminus \{0\}$. Then,

$$\operatorname{Res}(f, n\pi) = \operatorname{Res}\left(\frac{\frac{1}{z^2}}{\sin z}, n\pi\right) =$$

$$= \left.\frac{\frac{1}{z^2}}{\cos z}\right|_{z=n\pi} = /\cos n\pi = (-1)^n/$$

$$= \frac{(-1)^n}{(n\pi)^2} = \frac{(-1)^n}{n^2\pi^2}$$

To compute the residue at $z=0$,

note that $\sin z = z - \dfrac{z^3}{6} + O(z^5)$, $z \to 0$

Therefore,

$$\frac{1}{z^2 \sin z} = \frac{1}{z^2\left(z - \frac{z^3}{6} + O(z^5)\right)} =$$

$$= \frac{1}{z^3} \frac{1}{1 - \frac{z^2}{6} + O(z^4)} =$$

$$= \frac{1}{z^3}\left(a_0 + a_1 z + a_2 z^2 + O(z^3)\right)$$

where $\frac{1}{1 - \frac{z^2}{6} + O(z^4)} = a_0 + a_1 z + a_2 z^2 + O(z^3)$

i.e. $\left(a_0 + a_1 z + a_2 z^2 + O(z^3)\right)\left(1 - \frac{z^2}{6} + O(z^4)\right) = 1$

This gives

$$a_0 + a_1 z + \left(a_2 - \frac{a_0}{6}\right) z^2 + O(z^3) = 1$$

Thus, $a_0 = 1$, $a_1 = 0$, $a_2 = \frac{1}{6}$

Hence,

$$\frac{1}{z^2 \sin z} = \frac{1}{z^3}\left(1 + \frac{1}{6} z^2 + O(z^3)\right)$$

$$= \frac{1}{z^3} + \frac{1}{6}\frac{1}{z} + O(1), \quad z \to 0$$

Therefore,

$$\operatorname{Res}(f, 0) = \frac{1}{6}$$

Applying the residue thm, we get that

$$\int_{C_N} f(z)\,dz = 2\pi i\left(\frac{1}{6} + 2\sum_{n=1}^{N}\frac{(-1)^n}{n^2\pi^2}\right)$$

$n>0$ and $-c>\ldots$

$\square$

<u>Alt</u>: $\operatorname{Res}(f,0)$ can also be found as follows:

$$\frac{1}{z^2 \sin z} = \frac{1}{z^3}\,\frac{1}{1-\frac{z^2}{6}+O(z^4)} = \left/ w = \frac{z^2}{6}+O(z^4)\right/$$

$$= \frac{1}{z^3}\,\frac{1}{1-w} = \frac{1}{z^3}\left(1+w+O(w^2)\right) =$$

$$= \frac{1}{z^3}\left(1+\frac{z^2}{6}+O(z^4)\right) =$$

$$= \frac{1}{z^3} + \frac{1}{6}\,\frac{1}{z} + O(z), \quad z\to 0$$

Another way is as follows:

$$\frac{1}{z^2\sin z} = \frac{1}{z^3 g(z)}, \text{ where}$$

$$g(z) = 1 - \frac{z^2}{6} + \ldots$$

$$\Rightarrow \operatorname{Res}(f,0) = \lim_{z\to 0}\frac{1}{2!}\frac{d^2}{dz^2}\left(z^3 f(z)\right) =$$

$$= \frac{1}{2!}\frac{d^2}{dz^2}\frac{1}{g(z)}\bigg|_{z=0}$$

Now, clearly

$$\frac{d}{dz}\frac{1}{g(z)} = -\frac{g'(z)}{(g(z))^2}$$

and

$$\frac{d^2}{dz^2}\frac{1}{g(z)} = -\frac{g''(z)(g(z))^2 - g'(z)\cdot 2g(z)g'(z)}{(g(z))^4}$$

$$= \frac{2g(z)(g'(z))^2 - g''(z)(g(z))^2}{(g(z))^4}$$

$$\Rightarrow \text{Res}(f,0) = \frac{1}{2!}\cdot\frac{2\cdot 1\cdot 0^2 - \left(-\frac{1}{3}\right)\cdot 1^2}{1^4} =$$

$$= \frac{1}{6} \text{, as above}$$

(b) That $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

follows immediately if we can show

that

$$\lim_{N \to \infty} \int_{C_N} \frac{1}{z^2 \sin z} \, dz = 0$$

Now, on the vertical segments of C_N,

we have that

$$|z| \geq \left(N + \tfrac{1}{2}\right)\pi ,$$

$$|\sin z| \geq \left|\sin\left(N+\tfrac{1}{2}\right)\pi\right| = 1$$

So, $|f(z)| \leq \frac{1}{\left(\left(N+\frac{1}{2}\right)\pi\right)^2}$ on the verticals

And on the horizontal segments of C_N,

we have

$$|z| \geq \left(N + \tfrac{1}{2}\right)\pi$$

$$|\sin z| \geq \left|\sinh\left(N+\tfrac{1}{2}\right)\pi\right| \geq \sinh\tfrac{3\pi}{2} \geq 1$$

for $N \geq 1$.

So, $|f(z)| \leq \frac{1}{\left(\left(N+\frac{1}{2}\right)\pi\right)^2}$ on the horizontals

Thus, $|f(z)| \leq \frac{1}{\left(\left(N+\frac{1}{2}\right)\pi\right)^2} \quad \forall z \in C_N$

By the ML-ineq., it follows that

$$\left| \int_{C_N} \frac{1}{z^2 \sin \pi z} \, dz \right| \leq$$

$$\leq \frac{1}{((N+\frac{1}{2})\pi)^2} \cdot \underset{\substack{\uparrow \\ \text{4 sides}}}{4} \cdot \underbrace{2(N+\tfrac{1}{2})\pi}_{\text{side of square}} =$$

$$= \frac{4}{((2N+1)\pi)^2} \cdot 4 \cdot (2N+1)\pi = \frac{16}{(2N+1)\pi}$$

$$\to 0 \quad , N \to \infty$$

$\boxed{}$

(8) $$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |f(z)| \leq \frac{1}{1-|z|}, \ |z|<1$$

By the generalized Cauchy integral formula

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz$$

Thus, for any $r \in (0,1)$,

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz$$

$$\Rightarrow |a_n| \underset{ML}{\leq} \frac{1}{2\pi} \cdot \frac{1}{r^{n+1}} \cdot \frac{1}{1-r} \cdot 2\pi r =$$

$$= \frac{1}{r^n(1-r)}$$

Choose r so as to minimize the RHS,

i.e. choose r so that $r^n(1-r)$ is

maximized. Put $\lambda(r) = r^n(1-r)$, $0<r<1$

Note that $\lambda(r) \longrightarrow 0$ as $r \to 0^+$ and $r \to 1^-$

$$\lambda'(r) = n r^{n-1} - (n+1) r^n = r^{n-1}(n-(n+1)r)$$

$$\lambda'(r) = 0 \iff r = \frac{n}{n+1} < 1$$

Note the sign change + to - of

$\lambda'(r)$ near $r = \frac{n}{n+1}$

$$\Rightarrow \lambda(r) \leq \lambda\left(\frac{n}{n+1}\right) =$$

$$= \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) = \frac{n^n}{(n+1)^{n+1}}, \quad 0<r<1$$

so that

$$|a_n| \leq \frac{(n+1)^{n+1}}{n^n}$$

$\blacksquare$