UPPSALA UNIVERSITET
Matematiska institutionen
M. Klimek

Prov i matematik
Kurs: 1MA022
2021-06-04

Complex Analysis

Writing time: 08:00–13:00.
Other than writing utensils, paper and a copy of the textbook *Complex Analysis* by Gamelin, no help materials are allowed. Each problem is worth maximum 5 points. For the grades 3, 4 and 5, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be clearly written and properly explained. Bonus points from the homework assignments will be added to your exam result.

---

1. Determine the set of all complex numbers z solving the equation $-4\sin^2 z = \mathrm{e}^{2\mathrm{i}z}$.

2. Suppose that $u(x,y)$ and $v(x,y)$ are harmonic functions in a domain $D \subset \mathbb{C}$, such that $u(x,y) = -v^2(x,y)$ for all $z = x + \mathrm{i}y \in D$. Show that $f(z) = u(x,y) + \mathrm{i}v(x,y)$ is analytic in D only if f is a constant function.

3. Let w be a complex number. Consider the principal branch of $(1+z)^w$, that is the function

$$f(z) = \mathrm{e}^{w\,\mathrm{Log}\,(1+z)}, \qquad z \in D(0,1).$$

Show that

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{w(w-1)(w-2)\cdot\ldots\cdot(w-(n-1))}{n!} z^n, \qquad z \in D(0,1).$$

If $0 < r < 1$, is the series uniformly convergent in the disc $\overline{D}(0,r)$?

4. Find the Laurent series expansion centred at the origin of the function

$$f(z) = \frac{1}{z+2} - \frac{1}{z^4} + \frac{1}{(z-3)^2}$$

which is convergent in the domain $A = \{z \in \mathbb{C} \,:\, 2 < |z| < 3\}$.

5. Let H denote the upper half-plane, that is $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Find a conformal transformation of the domain $L = H \setminus \overline{D}(1,1)$ onto H.

6. Use the residue theorem to calculate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+2)^2}\, dx.$$

7. Consider the function

$$f(z) = \frac{z^3-1}{z-1} - 4z^{17}, \qquad z \in \mathbb{C} \setminus \{1\}.$$

Show that the function f has a removable singularity at $z = 1$. Show also that f has 17 zeros inside the unit disc $D(0,1)$.

8. Suppose that $p(z)$ is a polynomial of degree $d > 1$ with real coefficients. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an analytic function such that we have the inequality

$$|f(z)| \leq p\left(\sqrt{|z|}\right)$$

for all $z \in \mathbb{C}$. Show that f is a complex polynomial of degree not exceeding $d/2$.

GOOD LUCK!

SOLUTIONS

1. We have to solve the equation

$$-4\left(\frac{\mathrm{e}^{\mathrm{i}z}-\mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}}\right)^2=\mathrm{e}^{2\mathrm{i}z},$$

that is

$$\mathrm{e}^{2\mathrm{i}z}-2+\mathrm{e}^{-2\mathrm{i}z}=\mathrm{e}^{2\mathrm{i}z},$$

which is the same as $\mathrm{e}^{-2\mathrm{i}z}=2$. Let $z=x+\mathrm{i}y$. Since $\mathrm{e}^{2y}=|\mathrm{e}^{-2\mathrm{i}z}|=2$, we must have $y=\ln\sqrt{2}$. We also need $\mathrm{e}^{-2\mathrm{i}x}=\cos(-2x)+\mathrm{i}\sin(-2x)=1$. So $x=k\pi$ where $k\in\mathbb{Z}$.

2. If f is analytic, then the Cauchy-Riemann equations hold: $u_x=v_y$ and $u_y=-v_x$. But since $u=-v^2$, we also have $u_x=-2vv_x$ and $u_y=-2vv_y$. Therefore $u_x=2vu_y=-4v^2v_y=-4v^2u_x$. So $u_x(1+4v^2)=0$ identically in D. Consequently $u_x\equiv 0$, and so $v_y\equiv 0$ because of the Cauchy-Riemann equations. Thus u depends only on y and v depends only on x. Since $u=-v^2$, the required conclusion follows.

3. This is the Taylor series expansion at because $f(0)=1$ and

$$f^{(n)}(0)=w(w-1)\cdot\ldots\cdot(w-n+1)(1+z)^{w-n}\Big|_{z=0}=w(w-1)\cdot\ldots\cdot(w-n+1).$$

Since the function is analytic in the unit disc, the convergence is uniform on compact subsets of the unit disc.

4. We have

$$f(z)=-\frac{1}{z^4}+\frac{1}{z+2}+\left(\frac{1}{3-z}\right)'.$$

Since $2<|z|$ in A, we have $|-2/z|<1$ and $|z/3|<1$ and we can use the geometric series formula:

$$\frac{1}{z+2}=\frac{1}{z}\cdot\frac{1}{1-(-2/z)}=\sum_{n=1}^{\infty}\frac{(-2)^{n-1}}{z^n}$$

and

$$\frac{1}{3-z}=\frac{1}{3}\cdot\frac{1}{1-z/3}=\sum_{n=0}^{\infty}3^{-(n+1)}z^n.$$

Hence, in A

$$\begin{aligned} f(z) &= -\frac{1}{z^4}+\sum_{n=1}^{\infty}\frac{(-2)^{n-1}}{z^n}-\left(\sum_{n=0}^{\infty}3^{-(n+1)}z^n\right)' \\ &= -\frac{1}{z^4}+\sum_{n=1}^{\infty}\frac{(-2)^{n-1}}{z^n}+\sum_{n=1}^{\infty}n3^{-(n+1)}z^{n-1}. \end{aligned}$$

5. For example we can use the standard approach to lunar domains (see the textbook or the lecture notes). Since the "horns" of the domains are 2 and 0 we use the transformation

$$z \longmapsto \frac{z-2}{z} = 1 + \frac{-2}{z}$$

to map the domain onto the wedge, which then can be mapped onto H. The above transformation consists of a few elementary transformations which modify the domain as follows:

- $z \mapsto 1/z$ maps L onto the wedge $\{\operatorname{Re} z < 1/2,\ \operatorname{Im} z < 0\}$;
- $z \mapsto 1 - 2z$ maps the last set onto the first quadrant.

Finally $z \mapsto z^2$ maps the first quadrant onto H.

6. If $R > 2$, the function

$$f(z) = \frac{e^{iz}}{(z^2+2)^2}$$

has only one singularity within the upper half of $\bar{D}(0, R)$, a pole of order 2 at $i\sqrt{2}$. Also if Γ_R denotes the upper semicircle of radius R and centre at 0 oriented counterclockwise, then in view of Jordan's lemma

$$\left|\int_{\Gamma_R} f(z)\, dz\right| \leq \frac{1}{(R^2-1)^2}\int_{\Gamma_R} |e^{iz}||dz| < \frac{\pi}{(R^2-1)^2} \to 0 \text{ as } R \to \infty.$$

Since

$$\operatorname{Res}[f, i\sqrt{2}] = \left(\frac{e^{iz}}{(z+i\sqrt{2})^2}\right)'\Bigg|_{z=i\sqrt{2}} = -\frac{ie^{-\sqrt{2}}}{16}(2+\sqrt{2}),$$

the integral is equal to

$$\frac{\pi e^{-\sqrt{2}}}{4}\left(1 + \frac{1}{\sqrt{2}}\right).$$

7. Clearly $f(z) = z^2 + z + 1 - 4z^{17}$ for $z \neq 1$. Hence we get the first claim. We apply Rouche's theorem to the unit disc $D(0, 1)$ and the functions

$$g(z) = z^2 + z + 1 \quad \text{and} \quad h(z) = -4z^{17}.$$

If $|z| = 1$, then

$$|g(z)| \leq 3 < 4 = |h(z)|,$$

and so $f(z)$ has 17 zeros in the unit disk, because $h(z)$ has 17 zeros there.

8. Let $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$ be the power series expansion of f around zero and let $p(z) = b_0 + b_1 z + \ldots + b_d z^d$. Let $M = \max\{|b_0|, |b_1|, \ldots, |b_d|\}$. By the Cauchy estimates if $n \in \mathbb{N}$, $n > d/2$ and $R > 1$ we have

$$|a_n| = \frac{|f^{(n)}(0)|}{n!} \leq \frac{\sup\{|f(z)| : |z| = R\}}{R^n} \leq \frac{(d+1)MR^{d/2}}{R^n} \to 0,$$

as $R \to \infty$. Hence $f(z) = a_0 + a_1 z + \ldots + a_m z^m$, where m is the integer part of $d/2$.