UPPSALA UNIVERSITET
Matematiska institutionen
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Prov i matematik
KandFy3, KandMa2 m.fl.
Complex Analysis
2022–05–24

Writing time: 08.00 – 13.00. Allowed aids: Writing materials. Each problem has a maximum credit of 5 points. Bonus points from the homework assignments will be added to your exam result. For the grades 3, 4 and 5 respectively, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be clearly written and properly explained.

1. Solve the equation

$$\sin z + \cos z = 2.$$

The answer should be given in the form $a + ib$, where a and b are real numbers.

2. Find all functions $f = u + iv$ which are analytic in $\mathbb{C}$ and have imaginary part of the form

$$v(x, y) = x^2 + \phi(y),$$

where ϕ is a twice differentiable real-valued function. The answer should be given as an expression in the variable $z = x + iy$.

3. Find a Möbius transformation which maps the real line onto the circle $|w - 1| = 2$, the point i to the point 1, and the point -1 to itself.

4. Determine the Laurent series of the following functions in the regions indicated:

(a) $f(z) = \dfrac{3z}{z^2 + z - 2}$, in $1 < |z| < 2$,

(b) $g(z) = \operatorname{Log}\left(\dfrac{z+1}{z}\right)$, in $|z| > 1$.

5. Calculate the value of the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2 + 1)^2}\, dx.$$

6. Determine the number of solutions of the equation

$$z + 3 + 2e^z = 0$$

in the left half-plane $\operatorname{Re} z < 0$.

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7. For $0 < a < 1$, evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx$$

by integrating the function $f(z) = \dfrac{e^{az}}{1+e^z}$ around the rectangle with vertices at $\pm R$ and $\pm R + i2\pi$.

8. Find the largest value of $|f''(0)|$ where f is analytic in the unit disk $|z| < 1$ and satisfies

$$|f(z)| \leq \frac{1}{1-|z|}, \qquad |z| < 1.$$

GOOD LUCK!