UPPSALA UNIVERSITET
Matematiska institutionen
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Prov i matematik
KandFy3, KandMa2 m.fl.
Complex Analysis
2022–08–20

Writing time: 08.00 – 13.00. Allowed aids: Writing materials. Each problem has a maximum credit of 5 points. For the grades 3, 4 and 5 respectively, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be clearly written and properly explained.

1. Solve the equation
$$\tan z = \frac{2}{\sqrt{3}} - i.$$
The answer should be given in the form $a + ib$, where a and b are real numbers.

2. Find all functions $f = u + iv$ which are analytic in $\mathbb{C}$ and satisfy
$$u + v = e^x \sin y.$$
The answer should be given as an expression in the variable $z = x + iy$.

3. Determine the image of the right half-plane $\{z : \operatorname{Re} z > 0\}$ under the mapping $z \mapsto \dfrac{\operatorname{Log} z - i\pi/2}{\operatorname{Log} z + i\pi/2}$.

4. Find the Laurent series expansion of the function
$$f(z) = \frac{1}{z+1} + \frac{1}{(z-2)^2}$$
in the annulus $1 < |z| < 2$.

5. Determine if there is a function f having a removable singularity, a pole respectively an essential singularity at 0, satisfying:

(a) $f(1/n) = (-1)^n$ for $n = 1, 2, 3, ...$
(b) $f(1/n) = n$ for $n = 1, 2, 3, ...$
(c) $f(1/n) = n(-1)^n$ for $n = 1, 2, 3, ...$

Motivate your answers by giving examples of such f or by explaining why the singularity cannot be of a certain type.

6. Calculate the integral
$$\int_0^\infty \frac{\sin x}{x(x^2+1)^2}\, dx.$$
$\Big[$Hint: Integrate $\dfrac{e^{iz}}{z(z^2+1)^2}$ around the boundary of a half-disk indented at $z = 0.\Big]$

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7. Determine the number of zeros of the polynomial

$$p(z) = z^7 + z^5 + z^4 + 3z^2 - 4$$

in the right half-plane $\operatorname{Re} z > 0$.

8. Show that if $p(z)$ is a polynomial of degree $n > 1$ having simple zeros at $z_1, z_2, \ldots, z_n$, then

$$\sum_{j=1}^{n} \frac{1}{p'(z_j)} = 0.$$

GOOD LUCK!