UPPSALA UNIVERSITET
Matematiska institutionen
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Prov i matematik
KandFy3, KandMa2 m.fl.
Complex Analysis
2023–04–04

Writing time: 08.00 – 13.00. Allowed aids: Writing materials. Each problem has a maximum credit of 5 points. For the grades 3, 4 and 5 respectively, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be clearly written and properly explained.

1. Solve the equation

$$\sin z + (2+i)\cos z = 2+i.$$

The answer should be given in the form $a+ib$, where a and b are real numbers.

2. Find a function $f = u + iv$ analytic in $\mathbb{C} \setminus (-\infty, 0]$ which has real part

$$u(x,y) = 3x^2y - y^3 - 2xy + \ln(x^2+y^2)$$

and satisfies $f(1) = 0$. The answer should be given as an expression in the variable $z = x+iy$.

3. Find a conformal mapping which maps the first quadrant $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$ onto the upper half-plane $\operatorname{Im} w > 0$, so that 0 maps to ∞ and $1+i$ maps to itself.

4. Find the Laurent series expansion of the function

$$f(z) = \frac{3z-3}{2z^2-5z+2}$$

in the annulus $\frac{1}{2} < |z-1| < 1$.

5. Calculate the integral

$$\int_0^{2\pi} \frac{1}{5+4\sin\theta}\, d\theta.$$

6. Determine the number of zeros of the polynomial

$$p(z) = z^9 + z^7 - z^4 + 16$$

in the right half-plane $\operatorname{Re} z > 0$.

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7. Calculate the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 - \pi^2)}\, dx.$$

8. The function f is analytic in the whole complex plane with the exception of a double pole at $z = 2$, and a simple pole at $z = 0$ with residue 1. Furthermore, f has a double zero at $z = 1$. Finally, it holds that $\lim_{z \to \infty} f(z) = 1$. Show that these conditions determine f uniquely, and find f.

GOOD LUCK!