Uppsala Universitet
Matematiska Institutionen

Tenta i matematik
Complex analysis, 1MA022
2023-05-31

Skrivtid: 8:00-13:00. Hjälpmedel: inga. För betygen 3, 4, 5 krävs minst 18, 25 resp. 32 p. Alla svar ska motiveras med lämpliga beräkningar eller med en hänvisning till lämplig teori.

Problem 1 (5 pt).
1) Show that

$$\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

where log is some branch of the logarithm. (Hint: solve $\tanh(w) = z$ for w)

2) Find all solutions of the equation $\tanh z = i$.

Problem 2 (5 pt).

Show that $\ln(x^2 + y^2)$ is harmonic, and find its harmonic conjugate.

Problem 3 (5 pt). Compute the integral

$$\int_{-\pi}^{\pi} \frac{dx}{2 - (\cos x + \sin x)}.$$

(Hint: use the substitution $z = e^{ix}$.)

Problem 4 (6 pt). Compute the integral

$$\int_0^\infty \frac{x\,dx}{x^5 + 1}.$$

(Hint: First, plot all the singularities of the integrand, and based on that, choose an appropriate integration contour along the boundary of a radial sector of an appropriate angle)

Problem 5 (4 pt). Find the image of the unit disk under the Möbius transformation

$$T(z) = \frac{iz + 3}{iz - 1}.$$

Problem 6 (6 pt). Let N be a positive integer. Consider the function $\frac{1}{z^2 \sin z}$.

1) What kind of singularity does it have at 0? Compute the residue there.

2) Where are other singularities (outside of 0)? What kind of singularities are they? Compute residues there.

3) Use the Residue Theorem to show that

$$\frac{1}{2\pi i}\int_{C_R} \frac{dz}{z^2 \sin z} = \frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}$$

for any $\pi N < R < \pi(N+1)$.

Problem 7 (5 pt). Subdivide (arbitrarily) the boundary of the unit disk in three equal thirds. Find a harmonic function u on the unit disk $\mathbb{D}$, such that u is equal to 1 on the first third of the boundary, 0 on the next third, and -1 on the last third.

Problem 8 (4 pt). Consider the function

$$g(z) = \frac{e^{\frac{i\pi z}{2}} - 1}{e^{\frac{i\pi z}{2}} + 1}$$

which maps the set

$$\Omega = \{z \in \mathbb{C} : -1 < \mathrm{Re}(z) < 1\}$$

to $\mathbb{D}$.

Let $f : \mathbb{D} \mapsto \mathbb{C}$ be analytic, satisfying $f(0) = 0$. Suppose that $|\mathrm{Re}(f(z))| < 1$ for all $z \in \mathbb{D}$. By considering the function $F = g \circ f$, prove that

$$|f'(0)| \leq \frac{4}{\pi}$$

(Hint: use one of the conclusions of the Schwarz lemma)