Uppsala Universitet
Matematiska Institutionen

Tenta i matematik
Complex analysis, 1MA022
2023-05-31

Skrivtid: 8:00-13:00. Hjälpmedel: inga. För betygen 3, 4, 5 krävs minst 18, 25 resp. 32 p. Alla svar ska motiveras med lämpliga beräkningar eller med en hänvisning till lämplig teori.

Problem 1 (5 pt).
1) Show that

$$\tanh^{-1} z = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right),$$

where log is some branch of the logarithm. (Hint: solve $\tanh(w) = z$ for w)

Solution

If $w = \tanh^{-1}(z)$, then

$$z = \tanh(w) = \frac{\sinh(w)}{\cosh(w)} = \frac{e^w - e^{-w}}{e^w + w^{-w}} = \frac{e^{2w} - 1}{e^{2w} + 1}$$

Thus

$$\begin{aligned} z(e^{2w} + 1) = e^{2w} - 1 \implies \\ e^{2w} = \frac{1+z}{1-z} \implies \\ w = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) \end{aligned}$$

Hence,

$$\tanh^{-1} z = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right),$$

2) Find all solutions of the equation $\tanh z = i$.

Solution. Applying the inverse hyperbolic tangent on both sides: $z = \tanh^{-1}(i)$. Using part 1):

$$\begin{aligned} \tanh^{-1}(i) &= \frac{1}{2} \ln\left(\frac{1+i}{1-i}\right) \\ &= \frac{1}{2} \ln(i) \\ &= \frac{1}{2} \ln(1) + \frac{i}{2} (\pi over 2 + 2\pi n) \\ &= i\left(\frac{\pi}{4} + \pi n\right). \end{aligned}$$

Problem 2 (5 pt).

Show that $u(x, y) = \ln(x^2 + y^2)$ is harmonic, and find its harmonic conjugate.

Solution. As a composition of ln and a polynomial, u is twice continuously differentiable, and

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} \implies \frac{\partial^2 u}{\partial x^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2},$$
$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2} \implies \frac{\partial^2 u}{\partial y^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2},$$

therefore
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial 2u}{\partial y^2} = 0,$$
and the function is harmonic.

A harmonic conjugate v of u satisfies the Cauchy-Riemann equations
$$\begin{aligned} u_x &= v_y \\ v_x &= -u_y. \end{aligned}$$
We have therefore
$$v(x,y) = \int u_x dy + g(x) = \int \frac{2x}{x^2+y^2} dy + g(x) = 2\mathrm{arctan}\frac{y}{x} + g(x)$$
and
$$v(x,y) = -\int u_y dx + f(y) = -\int \frac{2y}{x^2+y^2} dx + f(y) = -2\mathrm{arctan}\frac{x}{y} + f(y) = 2\mathrm{arctan}\frac{y}{x} + \frac{\pi}{2} + f(y)$$
Comparing the two expression for v, we get
$$v(x,y) = 2\mathrm{arctan}\frac{y}{x} + c.$$

Problem 3 (5 pt). Compute the integral
$$\int_{-\pi}^{\pi} \frac{dx}{2-(\cos x + \sin x)}.$$
(Hint: use the substitution $z = e^{ix}$.)

Solution
$$\begin{aligned} \int_{-pi}^{\pi} \frac{dx}{2-(\cos x+\sin x)} &= \int_{-pi}^{\pi} \frac{dx}{2-(e^{ix}+e^{-ix})/2-(e^{ix}-e^{-ix})/(2i)} \\ &= \int_{|z|=1} \frac{-idz}{2z-(z^2+1)/2-(z^2-1)/(2i)} \\ &= (i-1)\int_{|z|=1} \frac{dz}{z^2-2(1+i)z+i} \\ &= (i-1)\int_{|z|=1} \frac{dz}{(z-z_+)(z-z_-)}, \end{aligned}$$
where
$$z_\pm = \left(1 \pm \frac{\sqrt{2}}{2}\right) + \left(1 \pm \frac{\sqrt{2}}{2}\right) i$$
Notice, that only z_- lies in the unit disk, therefore,
$$\begin{aligned} \int_{-pi}^{\pi} \frac{dx}{2-(\cos x+\sin x)} &= 2\pi i(i-1)\mathrm{Res}(f, z_-) \\ &= 2\pi i(i-1)\frac{1}{z_+ - z_-} \\ &= \sqrt{2}\pi \end{aligned}$$

Problem 4 (6 pt). Compute the integral

$$\int_0^\infty \frac{xdx}{x^5+1}.$$

(Hint: First, plot all the singularities of the integrand, and based on that, choose an appropriate integration contour along the boundary of a radial sector of an appropriate angle)

Solution. Consider the contour integral of $z/(z^5+1)$ along $L_R = [0,R]$, $C_R = \{z = Re^{it} : 0 \le t \le 2\pi/5\}$ and $M_R = \{te^{2\pi i/5} : 0 \le t \le R\}$. The only root of $z^5+1=0$ inside this contour is $z_0 = e^{\pi i/5}$. By Cauchy integral formula:

$$\begin{aligned}
\int_{L_R} \frac{zdz}{z^5+1} + \int_{C_R} \frac{zdz}{z^5+1} - \int_{M_R} \frac{zdz}{z^5+1} &= 2\pi i \text{Res}(f, z_0) \\
&= 2\pi i \lim_{z \to e^{\frac{\pi i}{5}}} \frac{(z - e^{\frac{\pi i}{5}})z}{(z^5+1)} \\
&= 2\pi i \frac{z}{(z-e^{\frac{3\pi i}{5}})(z+1)(z-e^{\frac{7\pi i}{5}})(z-e^{\frac{9\pi i}{5}})}\Big|_{z=e^{\frac{\pi i}{5}}} \\
&= 2\pi i \frac{e^{\frac{\pi i}{5}}}{(e^{\frac{\pi i}{5}}-e^{\frac{3\pi i}{5}})(e^{\frac{\pi i}{5}}+1)(e^{\frac{\pi i}{5}}-e^{\frac{7\pi i}{5}})(e^{\frac{\pi i}{5}}-e^{\frac{9\pi i}{5}})}.
\end{aligned}$$

For z on C_R:

$$\left|\frac{z}{z^5+1}\right| \le \frac{R}{R^5-1},$$

and

$$\left|\int_{C_R} \frac{zdz}{z^5+1}\right| \le \frac{2\pi}{5}\frac{R^2}{R^5-1},$$

which goes to 0 as $R \to \infty$. Next,

$$\begin{aligned}
\int_{M_R} \frac{zdz}{z^5+1} &= \int_{M_R} \frac{te^{\frac{2\pi i}{5}} dte^{\frac{2\pi i}{5}}}{t^5 e^{5\frac{2\pi i}{5}}+1} \\
&= e^{\frac{4\pi i}{5}} \int_{L_R} \frac{tdt}{t^5+1} \\
&= e^{\frac{4\pi i}{5}} \int_{L_R} \frac{tdt}{t^5+1}.
\end{aligned}$$

Therefore,

$$\left(1 - e^{\frac{4\pi i}{5}}\right) \int_0^\infty \frac{xdx}{x^5+1} = 2\pi i \frac{e^{\frac{\pi i}{5}}}{(e^{\frac{\pi i}{5}}-e^{\frac{3\pi i}{5}})(e^{\frac{\pi i}{5}}+1)(e^{\frac{\pi i}{5}}-e^{\frac{7\pi i}{5}})(e^{\frac{\pi i}{5}}-e^{\frac{9\pi i}{5}})},$$

therefore,

$$\int_0^\infty \frac{xdx}{x^5+1} = 2\pi i \frac{e^{\frac{\pi i}{5}}}{(e^{\frac{\pi i}{5}}-e^{\frac{3\pi i}{5}})(e^{\frac{\pi i}{5}}+1)(e^{\frac{\pi i}{5}}-e^{\frac{7\pi i}{5}})(e^{\frac{\pi i}{5}}-e^{\frac{9\pi i}{5}})(1-e^{\frac{4\pi i}{5}})}.$$

This can be left as the answer, or simplified to

$$\int_0^\infty \frac{xdx}{x^5+1} = \frac{\pi}{5}\sqrt{2 - \frac{2}{\sqrt{5}}}.$$

Problem 5 (4 pt). Find the image of the unit disk under the Möbius transformation

$$T(z) = \frac{iz+3}{iz-1}.$$

Solution. A Möbius transformation maps circles to circles or lines. We check three points on the unit circle:

$$T(i) = \frac{i^2+3}{i^2-1} = -1,$$
$$T(1) = \frac{i+3}{i-1} = -1-2i,$$
$$T(-i) = \frac{-i^2+3}{-i^2-1} = \infty,$$

thus, the images lie on a line $\{x = -1\}$. Furthermore, $T(0) = -3$, therefore the unit disk is mapped to the half plane $\{x < -1\}$.

Problem 6 (6 pt). Let N be a positive integer. Consider the function $\frac{1}{z^2 \sin z}$.

1) What kind of singularity does it have at 0? Compute the residue there.

Solution. At 0 $\sin z \approx z$, therefore, the function has a pole of order 3.

2) Where are other singularities (outside of 0)? What kind of singularities are they? Compute residues there.

Solution. $\sin(z) = 0$ for $z = \pi n$, and in a neighborhood of $z = \pi n$, $\sin(z) = \cos(\pi n)(z - \pi n) + O((z-\pi n)^2)$. Therefore, the function has a pole of order 1 at $z = \pi n$.

3) Use the Residue Theorem to show that

$$\frac{1}{2\pi i}\int_{C_R} \frac{dz}{z^2 \sin z} = \frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}$$

for any $\pi N < R < \pi(N+1)$.

Solution. We have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{C_R} \frac{dz}{z^2 \sin z} &= \frac{1}{2\pi i}\int_{C_R} \frac{dz}{z^2 \sin z} \\
&= \sum_{n=-N}^{N} \operatorname{Res}\left(\frac{1}{z^2 \sin(z)}, \pi n\right) \\
&= \operatorname{Res}\left(\frac{1}{z^2 \sin(z)}, 0\right) + \sum_{n\neq 0, n=-N}^{N} \frac{1}{\pi^2 n^2 \cos(\pi n)} \\
&= \operatorname{Res}\left(\frac{1}{z^2 \sin(z)}, 0\right) + 2\sum_{n=1}^{N} \frac{1}{\pi^2 n^2 \cos(\pi n)}.
\end{aligned}$$

What remains to compute is the residue at 0:

$$\begin{aligned}
\mathrm{Res}\left(\frac{1}{z^2\sin(z)},0\right) &= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\frac{z^3}{z^2\sin(z)}\\
&= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\frac{z}{\sin(z)}\\
&= \frac{1}{2}\lim_{z\to 0}\csc(z)(z\cot^2(z)-2\cot(z)+z\csc^2(z))\\
&= \frac{1}{2}\lim_{z\to 0}\frac{1}{\sin(z)}\left(\frac{z\cos^2(z)}{\sin^2(z)}-\frac{2\cos(z)}{\sin(z)}+\frac{z}{\sin^2(z)}\right)\\
&= \frac{1}{2}\lim_{z\to 0}\frac{z\cos^2(z)-2\cos(z)\sin(z)+z}{\sin^3(z)}\\
&= \frac{1}{2}\lim_{z\to 0}\frac{z(1-z^2/2)^2-2(1-z^2/2)(z-z^3/6)+z}{z^3}\\
&= \frac{1}{2}\lim_{z\to 0}\frac{z^3/3+z^5/12}{z^3}\\
&= \frac{1}{6}
\end{aligned}$$

Answer:

$$\frac{1}{2\pi i}\int_{C_R}\frac{dz}{z^2\sin z}=\frac{1}{6}+2\sum_{n=1}^{N}\frac{(-1)^n}{n^2\pi^2}.$$

Problem 7 (5 pt). Subdivide (arbitrarily) the boundary of the unit disk in three equal thirds. Find a harmonic function u on the unit disk $\mathbb{D}$, such that u is equal to 1 on the first third of the boundary, 0 on the next third, and -1 on the last third.

Solution. We know that $\arg(z)$ is equal to $-\pi$ on the negative real axis, π on the positive. Similarly, $\arg(z-1)$ is equal to $-\pi$ on the real semi axis $\{x<1\}$ and π on $\{x>1\}$. We arrange for $a\arg(z)+b\arg(z-1)+c$ to be equal to -1 on the negative real axis, to 0 on the interval $(0,1)$ and 1 for $x>1$:

$$\begin{aligned}
-a\pi-b\pi+c&=-1\\
a\pi-b\pi+c&=0\\
a\pi+b\pi+c&=1,
\end{aligned}$$

which gives $a=b=1/(2\pi)$.

Now, divide the boundary of the unit circle into three parts by points 1, $e^{2\pi i/3}$ and $e^{4\pi i/3}$. Let us find a Möbius transformation $T(z)=(az+b)/(z+c)$ which maps 1 to 0, $e^{2\pi i/3}$ to 1 and $e^{4\pi i/3}$ to ∞:

$$\begin{aligned}
(a+b)/(1+c)&=0\\
(ae^{2\pi i/3}+b)/(e^{2\pi i/3}+c)&=1\\
(ae^{4\pi i/3}+b)/(e^{4\pi i/3}+c)&=\infty.
\end{aligned}$$

The last equation implies

$$c=-e^{4\pi i/3}.$$

The remaining two have the solutions

$$a=-e^{\frac{2\pi i}{3}},\quad b=e^{\frac{2\pi i}{3}}.$$

Therefore the function $A\circ T$, where

$$A(z)=\frac{1}{2\pi}\arg(z)+\frac{1}{2\pi}\arg(z-1)$$

is harmonic on the unit disk, as a composition of a harmonic function A and a holomorphic T and is equal to -1, 0 and 1 on the corresponding thirds of the unit circle.

Problem 8 (4 pt). Consider the function

$$g(z) = \frac{e^{\frac{i\pi z}{2}} - 1}{e^{\frac{i\pi z}{2}} + 1}$$

which maps the set

$$\Omega = \{z \in \mathbb{C} : -1 < \mathrm{Re}(z) < 1\}$$

to $\mathbb{D}$.

Let $f : \mathbb{D} \mapsto \mathbb{C}$ be analytic, satisfying $f(0) = 0$. Suppose that $|\mathrm{Re}(f(z))| < 1$ for all $z \in \mathbb{D}$. By considering the function $F = g \circ f$, prove that

$$|f'(0)| \leq \frac{4}{\pi}$$

(Hint: use one of the conclusions of the Schwarz lemma)

Solution. Notice that $g(0) = 0$. The composition $F = g \circ f : \mathbb{D} \mapsto \mathbb{C}$ is analytic on $\mathbb{D}$, and since $|\mathrm{Re}(f(z)| < 1$, f maps $\mathbb{D}$ to Ω, we have that F maps $\mathbb{D}$ to itself and 0 to 0. By Schwarz Lemma $|F'(0)| \leq 1$. Since

$$F'(z) = g'(f(z))f'(z) = \frac{\pi i e^{\pi i f(z)/2} f'(z)}{(e^{\pi i f(z)/2} + 1)^2}$$

and $f(0) = 0$, we get

$$1 \geq |F'(0)| = \frac{\pi |f'(0)|}{4}$$

from which the result follows.