Uppsala Universitet
Matematiska Institutionen

Tenta i matematik
Complex analysis, 1MA022
2023-08-17

Skrivtid: 8:00-13:00. Hjälpmedel: inga. För betygen 3, 4, 5 krävs minst 18, 25 resp. 32 p. Alla svar ska motiveras med lämpliga beräkningar eller med en hänvisning till lämplig teori.

Problem 1 (4 pt).
Find all solutions of the equation

$$5\cos z - 3i\sin z = 2.$$

Solution.

$$\begin{aligned} 5\frac{e^{iz}+e^{-iz}}{2} - 3i\frac{e^{iz}-e^{-iz}}{2i} &= 2 \implies \\ 5e^{iz}+5e^{-iz}-3e^{iz}+3e^{-iz} &= 4 \implies \\ 2e^{iz}+8e^{-iz} &= 4 \implies \\ e^{2iz}-2e^{iz}+4 &= 0. \end{aligned}$$

Set $x = e^{iz}$, then we need to solve the quadratic equation

$$x^2 - 2x + 4 = 0 \implies x = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}.$$

Thus

$$e^{iz} = 1 \pm i\sqrt{3} \implies iz = \mathrm{Log}(1 \pm i\sqrt{3}) + i2\pi k = \ln|1 \pm \sqrt{3}| + i\mathrm{Arg}(1 \pm i\sqrt{3}) + i2\pi k = \ln 2 \pm i\frac{\pi}{3} + i2\pi k,$$

therefore,

$$z = -i\ln 2 \pm \frac{\pi}{3} + 2\pi k,$$

where k is any integer.

Problem 2 (5 pt).
Show that $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic, and find its harmonic conjugate.

Solution. As a polynomial, u is twice continuously differentiable, and

$$\begin{aligned} \frac{\partial u}{\partial x} &= 6xy + 4x \implies \frac{\partial^2 u}{\partial x^2} = 6y + 4, \\ \frac{\partial u}{\partial y} &= 3x^2 - 3y^2 - 4y \implies \frac{\partial^2 u}{\partial y^2} = -6y - 4, \end{aligned}$$

therefore

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial 2u}{\partial y^2} = 0,$$

and the function is harmonic.

A harmonic conjugate v of u satisfies the Cauchy-Riemann equations

$$\begin{aligned} u_x &= v_y \\ v_x &= -u_y. \end{aligned}$$

We have therefore

$$v(x,y) = \int u_x dy + g(x) = \int (6xy + 4x) dy + g(x) = 3xy^2 + 4xy + g(x)$$

and

$$v(x,y) = -\int u_y dx + f(y) = \int (3x^2 - 3y^2 - 4y)dx + f(y) = -x^3 + 3y^2x + 4yx + f(y).$$

Comparing the two expression for v, we get

$$v(x,y) = -x^3 + 3y^2x + 4yx + c.$$

Problem 3 (5 pt).
Let C be the circle $|z| = 1$ oriented counter-clockwise.

1) Compute

$$\int_C \frac{dz}{z^2 - 8z + 1}$$

Solution.

$$\frac{1}{z^2 - 8z + 1} = \frac{1}{(z - 4 - \sqrt{15})(z - 4 + \sqrt{15})}.$$

The only singularity inside the unite circle is $z = 4 - \sqrt{15}$. Therefore,

$$\begin{aligned}
\int_{|z|=1} \frac{dz}{z^2 - 8z + 1} &= 2\pi i \text{Res}(f, 4 - \sqrt{15}) \\
&= 2\pi i \lim_{z \to 4 - \sqrt{15}} \frac{(z - 4 + \sqrt{15})}{(z - 4 - \sqrt{15})(z - 4 + \sqrt{15})} \\
&= 2\pi i \frac{1}{z - 4 - \sqrt{15}}|_{z = 4 - \sqrt{15}} \\
&= -\frac{\pi i}{\sqrt{15}}.
\end{aligned}$$

2) Use part 1) to compute

$$\int_0^\pi \frac{d\theta}{4 - \cos\theta}$$

Solution.

$$\begin{aligned}
\int_{|z|=1} \frac{dz}{z^2 - 8z + 1} &= \int_{-\pi}^{\pi} \frac{de^{i\theta}}{e^{2i\theta} - 8e^{i\theta} + 1} \\
&= i \int_{-\pi}^{\pi} \frac{e^{i\theta} d\theta}{e^{2i\theta} - 8e^{i\theta} + 1} \\
&= i \int_{-\pi}^{\pi} \frac{d\theta}{e^{i\theta} + e^{-i\theta} - 8} \\
&= -\frac{i}{2} \int_{-\pi}^{\pi} \frac{d\theta}{4 - \cos\theta} \\
&= -i \int_0^{\pi} \frac{d\theta}{4 - \cos\theta}.
\end{aligned}$$

Therefore,

$$\int_0^\pi \frac{d\theta}{4 - \cos\theta} = \frac{\pi}{\sqrt{15}}.$$

Problem 4 (5 pt).

Let C_R be the circle $|z| = R$ $(R > 1)$ oriented counter-clockwise, and let Log denote the principle branch of logarithm.

1) Show that

$$\left|\int_{C_R} \frac{\mathrm{Log}(z^2)}{z^2}dz\right| < 4\pi\left(\frac{\mathrm{const}+\ln R}{R}\right)$$

Solution.

$$|\mathrm{Log}(z^2)| = |\ln|z|^2 + iArg(z^2)| \leq 2\ln|z| + \pi \leq 2\ln R + \pi.$$

Therefore,

$$\left|\int_{C_R} \frac{\mathrm{Log}(z^2)}{z^2}dz\right| < \int_0^{2\pi} \frac{2\ln R+\pi}{R^2}Rd\theta = 2\pi\frac{2\ln R+\pi}{R} = 4\frac{\ln R+\pi/2}{R}. \tag{1}$$

2) What can be said about the value of this integral as $R \to \infty$?

Solution. Since $\ln R$ grows much slower than R as $R \to \infty$, the integral (1) goes to zero.

Problem 5 (5 pt).

1) Show that all the roots of $z^{2011} + z^{2010} + z^{2009} + 1 = 0$ lie inside $|z| < 2$.

Solution.

$$z^{2011} + z^{2010} + z^{2009} + 1 = 0 \implies 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^{2011}} = 0.$$

If $|z| \geq 2$, then

$$\left|1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^{2011}}\right| \geq 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{2^{2011}} = \frac{1}{4} - \frac{1}{2^{2011}} > 0$$

and, thus, this region can not contain any zeros of the polynomial $z^{2011} + z^{2010} + z^{2009} + 1$.

2) Compute the integral

$$\int_C \frac{z^{2011}}{z^{2011} + z^{2010} + z^{2009} + 1}dz$$

where C is the circle $|z| = 2$.

Solution.

$$\begin{aligned}\frac{z^{2011}}{z^{2011} + z^{2010} + z^{2009} + 1} &= 1 - \frac{1}{z} + \left(\frac{z^{2011}}{z^{2011} + z^{2010} + z^{2009} + 1} - \frac{(1+1/z)(z^{2011} + z^{2010} + z^{2009} + 1)}{z^{2011} + z^{2010} + z^{2009} + 1}\right) \\ &= 1 - \frac{1}{z} + \frac{z^{2009} - z - 1}{z^{2011} + z^{2010} + z^{2009} + 1}.\end{aligned}$$

Notice that

$$\begin{aligned}I = \int_{|z|=R} \left|\frac{z^{2009} - z - 1}{z^{2011} + z^{2010} + z^{2009} + 1}\right| |dz| &\geq \int_0^{2\pi} \frac{R^{2009} + R + 1}{R^{2011} - R^{2010} - R^{2009} - 1}Rd\theta \\ &= 2\pi\frac{R^{2010} + R^2 + R}{R^{2011} - R^{2010} - R^{2009} - 1},\end{aligned}$$

and

$$\lim_{R\to\infty} I = \lim_{R\to\infty} \frac{R^{2010}}{R^{2011}} = 0. \tag{2}$$

Therefore, all zeros of the denominator are in inside the circle of radius 2, the integral of the original fraction over

$|z| = 2$ is the same as the integral of over any circle $|z| = R$ for $R > 2$:

$$\int_{|z|=R} \frac{z^{2011}}{z^{2011}+z^{2010}+z^{2009}+1} dz = \int_{|z|=R} 1 - \frac{1}{z} dz + \int_{|z|=R} \frac{z^{2009}-z-1}{z^{2011}+z^{2010}+z^{2009}+1} dz.$$

The first integral above is equal to

$$\int_{|z|=R} \left(1 - \frac{1}{z}\right) dz = -2\pi i,$$

by Cauchy integral theorem, while the second goes to zero as $R \to \infty$, according to (2).

Answer: $-2\pi i$.

Problem 6 (5 pt). Let $\mathbb{C}^*$ be the complex plane punctured at 0. Suppose $f : \mathbb{C}^* \mapsto \mathbb{C}$ is analytic in $\mathbb{C}^*$ with a pole of order 1 at 0. Additionally, suppose $f(z) \in \mathbb{R}$ for all $|z| = 1$. Show that for all $z \in \mathbb{C}^*$,

$$f(z) = \alpha z + \bar{\alpha}\frac{1}{z} + \beta,$$

for some $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{R}$.

Solution. Since f has a pole of order 1 at 0 and is analytic otherwise, the Laurent series expansion of f takes the form

$$f(z) = \sum_{n=-1}^{\infty} a_n z^n, \tag{3}$$

where the coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^{n+1}} dz \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta. \end{aligned}$$

Since $f(e^{i\theta}) = \overline{f}(e^{i\theta})$,

$$\begin{aligned} a_n &= \frac{1}{2\pi} \int_0^{2\pi} \overline{f}(e^{i\theta}) e^{-in\theta} d\theta \\ &= \frac{1}{2\pi} \overline{\int_0^{2\pi} f(e^{i\theta}) e^{in\theta} d\theta} \\ &= \bar{a}_{-n}. \end{aligned}$$

Since $a_{-n} = 0$ for all $n > 1$, we get that $a_n = 0$ for all $n > 1$, and the only terms surviving are

$$f(z) = a_1 z + \bar{a}_1 \frac{1}{z} + a_0.$$

Also, the equality $a_n = a_{\bar{-}n}$ for $n = 0$ implies that $a_0 = \beta \in \mathbb{R}$. Since $f(z)$ has a pole of order 1 at 0, a_1 and $\bar{a}_1$ are non-zero.

Problem 7 (5 pt).

Subdivide (arbitrarily) the boundary of the unit disk in two equal parts. Find a harmonic function u on the unit disk $\mathbb{D}$, such that u is equal to 1 on one half of the boundary and 0 on the other.

Solution. The argument function $\mathrm{Arg}(z)$ is harmonic in the upper half plane, $\mathrm{Arg}(z) = \pi$ on the positive real axis, and $\mathrm{Arg}(z) = -\pi$ on the negative real axis. Therefore,

$$T(z) = \frac{1}{2\pi}\mathrm{Arg}(z) + \frac{1}{2}$$

maps the positive real axis to 1, and the negative real axis to 0.

Now, consider the transformation

$$W(z) = \frac{z-i}{z+i},$$
$$W^{-1}(w) = i\frac{1-w}{1+w}$$

which maps the upper half plane $\mathbb{H}$ to the unit disk $\mathbb{D}$. Specifically, $W(0) = -1$ and $W(\infty) = 1$. W maps the positive real axis to the lower semi-circle, the negative - to the positive semi-circle.

Therefore the map $T(W^{-1}(w))$ is harmonic on the unit disk, and maps the lower semi-circle to 1, the upper - to 0.

Problem 8 (6 pt). Let $f : \mathbb{H} \mapsto \mathbb{H}$ be a holomorphic map of the upper half-plane into itself. Prove that for every $a \in \mathbb{H}$ we have

$$|f'(a)| \leq \frac{\mathrm{Im}(f(a))}{\mathrm{Im}(a)}$$

(*Hint: Precompose and postcompose with a map from a unit disk to* $\mathbb{H}$ *that sends* 0 *to* a *and* $h(a)$ *respectively, and use Schwarz lemma.*)

Solution. We have that the transformation

$$W(z) = \frac{z-i}{z+i}$$

maps the upper half plane to the unit disk. In particular, it maps point a to $\alpha = W(a) = (a-i)/(a+i)$. Further,

$$T_\alpha(w) = \frac{w-\alpha}{1-\bar{\alpha}w}$$

maps point α to zero. Therefore, the map $T_\alpha \circ W$ maps the upper half plane to $\mathbb{D}$ and point a to zero. Similarly, the map $T_\beta \circ W$, $\beta = (f(a)-i)/(f(a)+i)$ maps the upper half plane to $\mathbb{D}$ and point $f(a)$ to zero. Therefore, the map

$$F = T_\beta \circ W \circ f \circ (T_\alpha \circ W)^{-1}$$

maps the unit disk to itself and fixes zero. By Schwarz lemma:

$$\begin{aligned}
|F'(0)| \leq 1 &\implies |(T_\beta \circ W)'(f(a))||f'(a)||T_\alpha \circ W)'(a)|^{-1} \leq 1 \\
&\implies |f'(a)| \leq \frac{|T_\alpha \circ W)'(a)|}{|(T_\beta \circ W)'(f(a))|}.
\end{aligned}$$

We have

$$\begin{aligned}
W'(z) &= \frac{2i}{(z+i)^2} \implies W'(a) = \frac{2i}{(a+i)^2} \\
T'_\alpha(w) &= \frac{1-|\alpha|^2}{(1-\bar{\alpha}w)^2} \implies T'_\alpha(\alpha) = \frac{1-|\alpha|^2}{(1-|\alpha|^2)^2} = \frac{1}{1-|\alpha|^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|T'_\alpha(\alpha)W'(a)| &= \frac{2}{|a+i|^2}\frac{1}{1-|\alpha|^2} \\
&= \frac{2}{(a+i)(\bar{a}-i)}\frac{1}{1-\frac{(a-i)(\bar{a}+i)}{(a+i)(\bar{a}-i)}} \\
&= \frac{2}{(a+i)(\bar{a}-i)-(a-i)(\bar{a}+i)} \\
&= \frac{2}{|a|^2+i\bar{a}-ia+1-|a|^2+i\bar{a}-ia-1} \\
&= \frac{1}{2\mathrm{Im}(a)}.
\end{aligned}$$

A similar calculation holds for $|(T_\beta \circ W)'(f(a))|$:

$$|(T_\beta \circ W)'(f(a))| = \frac{1}{2\mathrm{Im}(f(a))}.$$

Therefore,

$$|f'(a)| \leq \frac{|T_\alpha \circ W)'(a)|}{|(T_\beta \circ W)'(f(a))|} = \frac{\mathrm{Im}(f(a))}{\mathrm{Im}(a)}.$$