

Skrivtid: 8:00-13:00. Hjälpmedel: inga. För betygen 3, 4, 5 krävs minst 18, 25 resp. 32 p. Alla svar ska motiveras med lämpliga beräkningar eller med en hänvisning till lämplig teori.

Problem 1 (4 pt). Find all solutions of the equation

$$3 \sin z + 5i \cos z = 2.$$

Problem 2 (5 pt). Show that $u(x, y) = \sinh(x) \cos(y)$ is harmonic, and find its harmonic conjugate.

Problem 3 (6 pt). Compute

$$\int_0^{2\pi} \frac{\sin \theta d\theta}{2 + \sin \theta}$$

using complex integration.

Problem 4 (6 pt).

1) How many solutions does the equation $e^z - 2z = 0$ have inside the circle $|z| = 3$? (use the Argument principle).

2) Using Rouché's theorem, show that the polynomial $P(z) = z^5 + 14z + 2$ has 4 roots in the annulus $3/2 < |z| < 2$.

Problem 5 (4 pt). Compute

$$\int_C \frac{ze^z}{z^2 + 1} dz$$

where $C = |z| = 2$ is oriented counterclockwise.

Problem 6 (5 pt). Find the Laurent series expansion centered at 0 of

$$\frac{1}{z(z-1)(z-2)}$$

in three different regions (which ones?).

Problem 7 (5 pt). Subdivide (arbitrarily) the boundary of the disk $|z - 1| \leq 2$ in two equal parts. Find a harmonic function u on this disk, such that u is equal to 1 on one half of the boundary and -1 on the other.

Problem 8 (5 pt). Write out both statements of the Schwarz lemma for a holomorphic function on the upper half plane $f : \mathbb{H} \mapsto \mathbb{H}$. (*Hint:* the upper half plane is conformally isomorphic to a unit disk).