# Final Exam – Complex Analysis, 1MA022

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duration of the exam: 5 hours

- Please write your answers in **English.**
- There are 8 problems in this exam, and each one is worth 5 points. The grade limits are: 18 points for grade 3, 25 points for grade 4 and 32 points for grade 5.
- For every question, you need to **justify every step** in your solution to get the full score.

**Good luck!!**

1. Let $D = \{z \in \mathbb{C} \colon |z - i| < 2\}$ and denote the boundary of $D$ by $\partial D$. Find a solution to the Dirichlet problem on $D = \{z \in \mathbb{C} \colon |z - i| < 2\}$ with boundary condition

$$\phi(z) = \begin{cases} 2 & \text{if } z = x + iy \in \partial D \text{ and } y > 1 \\ 1 & \text{if } z = x + iy \in \partial D \text{ and } y < 1 \end{cases}.$$

Hint: You can use the fact that the function

$$\psi(z) = \frac{A - B}{\pi} \operatorname{Arg}(z) + \frac{A + B}{2}$$

solves the Dirichlet problem on $\{z = x + iy \in \mathbb{C} \colon x > 0\}$ with boundary condition

$$\psi(iy) = \begin{cases} A & \text{if } y \in \mathbb{R} \text{ and } y > 0 \\ B & \text{if } y \in \mathbb{R} \text{ and } y < 0 \end{cases}.$$

2. Let

$$f(z) = \frac{2}{z^2 - 3z - 4}.$$

Find the Laurent series expansion of $f$ centered at 2 in the region $A = \{z \in \mathbb{C} \colon 2 < |z - 2| < 3\}$.

3. Decide if each of the statements below is TRUE or FALSE. Justify all your answers.
   (a) There exists a holomorphic function $f \colon \mathbb{C} \to \mathbb{C}$ whose image is equal to the set $S = \{z = x + iy \in \mathbb{C} \colon y \geq 0\}$.
   (b) Let $A = \{z \in \mathbb{C} \colon 0 < |z| < 1\}$. There exists a holomorphic function $f \colon A \to \mathbb{C}$ such that $|f(z)| \leq 100$ for every $z \in A$ and $\lim_{z \to 0} f(z)$ does not exist.
   (c) If $D \subset \mathbb{R}^2$ is an open disk and $u \colon D \to \mathbb{R}$ is a harmonic function, then $u$ is infinitely differentiable on $D$.

4. Let

$$f(z) = \frac{e^z}{1 - z^2}.$$

For each $R > 0$, let $\gamma_R$ be the line segment in the imaginary axis that starts at $-iR$ and ends at $iR$. Compute

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)\,\mathrm{d}z.$$

5. Calculate the integral

$$\int_0^{2\pi} \frac{1}{3 - 2\sin\theta}\,\mathrm{d}\theta.$$

6. (a) State the Maximum Modulus Principle.
   (b) Use the Maximum Modulus Principle to prove the Fundamental Theorem of Algebra.

7. Consider the series

$$\sum_{k=0}^{\infty} z^k e^{-kz}.$$

Show that the series defines a holomorphic function $f(z)$ on the set $D = \{z \in \mathbb{C} \colon |z| < 1 \text{ and } \mathrm{Re}(z) > 0\}$.

Hint: Given $\epsilon > 0$, consider the domain $D_\epsilon = \{z = x + iy \in D \colon x > \epsilon\}$.

8. Show that the equation $3z^{13} = e^z$ has 13 **distinct** solutions in the disk $\{|z| < 1\} \subset \mathbb{C}$.