Re-exam – Complex Analysis, 1MA022

Uppsala University
Department of Mathematics
Luís Diogo

2024-08-22
duration of the exam: 5 hours

- Please write your answers in **English.**
- The total number of available points is 40. The grade limits are: 18 points for grade 3, 25 points for grade 4 and 32 points for grade 5.
- For every question, you need to **justify every step** in your solution to get the full score.

Good luck!!

1. [5 points] Find all the complex solutions of the equation
$$\sin(z) + 3i\cos(z) = 4.$$

2. [6 points] Let $D_1 = \{z \in \mathbb{C} \colon |z| < 1\}$ and $D_2 = \{z = x + iy \in \mathbb{C} \colon |z| < 1 \text{ and } y > 0\}$. Find a holomorphic bijection from D_2 to D_1.

3. [6 points] Let
$$f(z) = \frac{z}{1 - e^{-z}}.$$
Find all the singularities of f. Determine which singularities are removable, essential and poles. For each pole, find its order and compute its residue.

4. [6 points] Compute
$$\int_{-\infty}^{\infty} \frac{1}{1 + x^4}\,\mathrm{d}x.$$

5. [6 points] Find the number of zeros of the function $f(z) = z^4 + z^3 + z + 16$ in the positive quadrant $\{x + iy \in \mathbb{C} \colon x > 0 \text{ and } y > 0\}$.

6. [6 points]
   (a) State Rouché's Theorem.
   (b) Use Rouché's Theorem to prove the Fundamental Theorem of Algebra.

7. [5 points] Let $f \colon D(0,1) \to D(0,1)$ be a holomorphic function with a zero of order $m \geq 1$ at $z_0 = 0$. Show that for all $z \in D(0,1)$ we have
$$|f(z)| \leq |z|^m.$$