Re-exam – Complex Analysis, 1MA022

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duration of the exam: 5 hours

- Please write your answers in **English.**
- The total number of available points is 40. The grade limits are: 18 points for grade 3, 25 points for grade 4 and 32 points for grade 5.
- For every question, you need to **justify every step** in your solution to get the full score.
- In this exam, given $z_0 \in \mathbb{C}$ and $r > 0$, we denote the disk $\{z \in \mathbb{C} : |z - z_0| < r\}$ by $D(z_0, r)$.

Good luck!!

1. [**6 points in total**] Consider the function

$$f(z) = \frac{z^2 + 1}{z^2 - 4}.$$

   (a) [3 points] Find all the singularities of f. Determine which singularities are removable, essential and poles. For each pole, find its order and compute its residue.

   (b) [3 points] Determine the Laurent series expansion of f in the annulus $A = \{z \in \mathbb{C} : 5 < |z| < 6\}$.

2. [**8 points in total**] Consider the intersection

$$A = D(0,1) \cap \left\{ x + iy \in \mathbb{C} : y > \frac{\sqrt{2}}{2} \right\}$$

and the set

$$B = \left\{ re^{i\theta} \in \mathbb{C} : r > 0 \text{ and } 0 < \theta < \frac{\pi}{4} \right\}.$$

   (a) [3 points] Find a holomorphic bijection from B to $D(0,1)$?

   (b) [3 points] Find a holomorphic bijection from A to B.
   The fact that holomorphic maps are conformal may help with this question.

   (c) [2 points] Find a holomorphic bijection from A to $D(0,1)$?

3. [**6 points**] Compute

$$\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} \, \mathrm{d}x.$$

4. **[5 points]** Determine the number of zeros that the function $f(z) = z^5 + 10z + 1$ has in the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$, counting multiplicities.

5. **[6 points in total]**

   (a) [3 points] State the Schwarz Lemma.

   (b) [3 points] Use the Schwarz Lemma to prove the following: if $f \colon D(0,1) \to D(0,1)$ is a holomorphic bijection such that $f(0) = 0$, then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $f(z) = \lambda z$ for all $z \in D(0,1)$.

6. **[9 points in total]** Let $a > 0$ be a constant positive real number, and let f be a holomorphic function on the strip $S = \{x + iy \in \mathbb{C} : \ -a < y < a\}$. Suppose that f is 2π-periodic, which means that $f(z + 2\pi) = f(z)$ for every $z \in S$. The Fourier coefficients of f are defined as

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, e^{-inx}\, \mathrm{d}x$$

   for every integer $n \in \mathbb{Z}$.

   (a) [3 points] Use the multivalued logarithm to show that there is a unique holomorphic function g on the annulus $A = \{z \in \mathbb{C} : e^{-a} < |z| < e^{a}\}$ such that $f(z) = g(e^{iz})$ for every $z \in S$.

   (b) [3 points] Relate the Fourier coefficients $\widehat{f}(n)$ with the Laurent series of g on A.

   (c) [3 points] Prove that there exist constants $M > 0$ and $r > 1$ such that

$$|\widehat{f}(n)| \leq M\, r^{-n}$$

   for all $n \in \mathbb{Z}$.

   Hint: Use the fact that the Laurent series of g converges absolutely in A.

   Note: One can actually find constants $M > 0$ and $r > 1$ such that $|\widehat{f}(n)| \leq M\, r^{-|n|}$ for all $n \in \mathbb{Z}$, but you do not have to prove it. This in an instance of an important result called Paley–Wiener Theorem.