UPPSALA UNIVERSITET
Matematiska institutionen
Jörgen Östensson

Prov i matematik
KandFy3, KandMa2 m.fl.
Complex Analysis
2025–06–04

Writing time: 08.00 – 13.00. Allowed aids: Writing materials. Each problem has a maximum credit of 5 points. Bonus points from the homework assignments will be added to your exam result. For the grades 3, 4 and 5 respectively, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be clearly written and properly explained.

1. Solve the equation

$$\cos z - \sin z = 2.$$

The answer should be given in the form $a + ib$, where a and b are real numbers.

2. Find all functions $f = u + iv$ which are analytic in $\mathbb{C}$ and have real part of the form

$$u(x, y) = \phi(x)\,(1 - y),$$

where ϕ is a real-valued function of one variable of class C^2, which moreover satisfy $f(0) = 0$. The answer should be given as an expression in the variable $z = x + iy$.

3. Determine the image of the first quadrant $\{z : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ under the mapping $f(z) = \dfrac{z^3 - 2i}{z^3 + 2i}$. Also, determine the image of the ray $z = re^{i\pi/4}$, $r \geq 0$.

4. Compute the values of the following integrals:

(a) $\displaystyle\int_\Gamma \frac{1}{z - i}\,dz$, where Γ is the line segment from 0 to $1 + 2i$.

(b) $\displaystyle\int_C \frac{e^z}{z(e^z - 1)}\,dz$, where C is the counterclockwise oriented circle $|z| = 1$.

5. Find the Laurent series expansion of the function

$$f(z) = \frac{1}{z^2 + 3z + 2}$$

in the annulus $2 < |z - 1| < 3$.

6. Determine the number of zeros of the polynomial

$$p(z) = z^4 + z^3 + 6z^2 + 4z + 5$$

in the first quadrant $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$.

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7. Calculate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^4 - \pi^4}\, dx.$$

8. Determine all entire functions f such that $|f(z)| = 1$ for $|z| = 1$ and $f(z) \neq 0$ for $0 < |z| < 1$.

GOOD LUCK!