UPPSALA UNIVERSITET
Matematiska institutionen
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Prov i matematik
KandFy3, KandMa2 m.fl.
Complex Analysis
2025–06–04

Writing time: 08.00 – 13.00. Allowed aids: Writing materials. Each problem has a maximum credit of 5 points. Bonus points from the homework assignments will be added to your exam result. For the grades 3, 4 and 5 respectively, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be clearly written and properly explained.

1. Solve the equation
$$\cos z - \sin z = 2.$$
The answer should be given in the form $a + ib$, where a and b are real numbers.

2. Find all functions $f = u + iv$ which are analytic in $\mathbb{C}$ and have real part of the form
$$u(x, y) = \phi(x)\,(1 - y),$$
where ϕ is a real-valued function of one variable of class C^2, which moreover satisfy $f(0) = 0$. The answer should be given as an expression in the variable $z = x + iy$.

3. Determine the image of the first quadrant $\{z : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ under the mapping $f(z) = \dfrac{z^3 - 2i}{z^3 + 2i}$. Also, determine the image of the ray $z = re^{i\pi/4}$, $r \geq 0$.

4. Compute the values of the following integrals:

(a) $\displaystyle\int_\Gamma \frac{1}{z - i}\,dz$, where Γ is the line segment from 0 to $1 + 2i$.

(b) $\displaystyle\int_C \frac{e^z}{z(e^z - 1)}\,dz$, where C is the counterclockwise oriented circle $|z| = 1$.

5. Find the Laurent series expansion of the function
$$f(z) = \frac{1}{z^2 + 3z + 2}$$
in the annulus $2 < |z - 1| < 3$.

6. Determine the number of zeros of the polynomial
$$p(z) = z^4 + z^3 + 6z^2 + 4z + 5$$
in the first quadrant $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$.

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7. Calculate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^4 - \pi^4}\, dx.$$

8. Determine all entire functions f such that $|f(z)| = 1$ for $|z| = 1$ and $f(z) \neq 0$ for $0 < |z| < 1$.

GOOD LUCK!

Svar till tentamen i Complex Analysis 2025–06–04

1. $z = -\frac{\pi}{4} + 2n\pi \pm i\ln(\sqrt{2}+1)$, $n \in \mathbb{Z}$.

2. $f(z) = A\Big(z + \frac{iz^2}{2}\Big)$, $A \in \mathbb{R}$.

3. $\{w : \operatorname{Im} w > 0\} \cup \{w : |w| < 1\}$ resp. $\{w = -i + \sqrt{2}e^{i\theta}, \pi/4 < \theta \le 3\pi/4\}$.

4. (a) $\frac{\ln 2}{2} + i\frac{3\pi}{4}$, (b) πi.

5. $f(z) = \sum_{n=1}^{\infty}(-2)^{n-1}\frac{1}{(z-1)^n} + \sum_{n=0}^{\infty}\Big(-\frac{1}{3}\Big)^{n+1}(z-1)^n$.

6. 0.

7. $-\frac{1+e^{-\pi}}{2\pi}$.

8. $f(z) = az^N$, where $a \in \mathbb{C}$, $|a| = 1$, and N is a non-negative integer.