

Writing time: 14.00 – 19.00. Allowed aids: Writing materials. Each problem has a maximum credit of 5 points. Bonus points from the homework assignments will be added to your exam result. For the grades 3, 4 and 5 respectively, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be clearly written and properly explained.

1. Solve the equation

$$\tan z = -\frac{2}{\sqrt{3}} - i.$$

The answer should be given in the form $a + ib$, where a and b are real numbers.

2. Find all functions $f = u + iv$ which are analytic in \mathbb{C} and have real part of the form

$$u(x, y) = (x - y)\phi(x + y),$$

where ϕ is a real-valued function of one variable of class C^2 , which moreover satisfy $f(0) = 0$. The answer should be given as an expression in the variable $z = x + iy$.

3. Find a Möbius transformation which maps the real line onto the circle $|w - 1| = 2$, the point i to the point 1, and the point -1 to itself.

4. Use the residue theorem to calculate the integral

$$\int_0^{2\pi} \frac{\cos \theta}{2 + \cos \theta} d\theta.$$

5. The function

$$f(z) = \frac{(z^2 - \pi^2)(1 - \cos z)}{z^4 \sin z}$$

has a pole at the origin.

(a) Find the principal part of f at the origin, i.e. that part of the Laurent series of f in a punctured neighborhood of the origin which contains the negative powers of z .

(b) The Laurent series of f considered in (a) has region of convergence of the form $0 < |z| < R$. Determine R .

6. Determine the number of zeros of the function $z^2 + 2 + e^{iz}$ in the open square with vertices at 1, -1 , $1 + 2i$ and $-1 + 2i$.

Turn page!

7. Calculate the integral

$$\int_0^{\infty} \frac{\sqrt{x}}{x^2 + 5x + 6} dx.$$

8. Suppose f is entire and that there are constants $A, R > 0$ and a natural number N such that

$$|f(z)| \geq A|z|^N \quad \text{for } |z| \geq R.$$

Show that f must be a polynomial having degree at least N .

GOOD LUCK!