

TENTAMEN - LINEAR ALGEBRA II 2018/01/09

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1. (i) For which values of $a \in \mathbb{R}$ do the following polynomials form a basis $B = \{p_1(x), p_2(x), p_3(x)\}$ of $P_2(\mathbb{R})$:

$$p_1(x) = 2 + x^2$$

$$p_2(x) = (7 + a)x - 3x^2$$

$$p_3(x) = 4x + ax^2$$

Justify your answer.

- (ii) Let $B' = \{1, x, x^2\}$. In case that B is a basis provide the transition matrix $P_{B' \leftarrow B}$ from B to B' .

Possible solution 1a: (i) Since $\dim P_2(\mathbb{R}) = 3$, it suffices to determine when $p_1(x), p_2(x), p_3(x)$ are linearly independent. For this we have to check when

$$\lambda_1 p_1(x) + \lambda_2 p_2(x) + \lambda_3 p_3(x) = 0$$

has only the trivial solution $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Plugging in p_1, p_2 , and p_3 , we obtain

$$\lambda_1(2 + x^2) + \lambda_2((7 + a)x - 3x^2) + \lambda_3(4x + ax^2) = 0.$$

Comparing coefficients we see that this is the case if and only if

$$2\lambda_1 = 0$$

$$(7 + a)\lambda_2 + 4\lambda_3 = 0$$

$$\lambda_1 - 3\lambda_2 + a\lambda_3 = 0$$

We solve this system using Gaussian elimination.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 7+a & 4 \\ 1 & -3 & a \end{pmatrix} \xrightarrow{III - \frac{1}{2}I} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 7+a & 4 \\ 0 & -3 & a \end{pmatrix} \xrightarrow{II \leftrightarrow III} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & a \\ 0 & 7+a & 4 \end{pmatrix} \xrightarrow{III + \frac{7+a}{3}II} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & a \\ 0 & 0 & 4 + \frac{7+a}{3}a \end{pmatrix}$$

We see that this has only the trivial solution if and only if $4 + \frac{7+a}{3}a \neq 0$, i.e. if and only if $a^2 + 7a + 12 \neq 0$. With the help of the pq -formula it follows that the zeroes

of this equation are given by $a_{1/2} = -\frac{7}{2} \pm \sqrt{\frac{49}{4} - \frac{48}{4}} = \begin{cases} -4 \\ -3 \end{cases}$.

It follows that $p_1(x), p_2(x), p_3(x)$ form a basis if and only if $a \neq -4$ and $a \neq -3$.

(ii) Since

$$\begin{aligned} 2 + x^2 &= \mathbf{2} \cdot \mathbf{1} + \mathbf{0} \cdot x + \mathbf{1} \cdot x^2 \\ (7 + a)x - 3x^2 &= \mathbf{0} \cdot \mathbf{1} + (\mathbf{7} + \mathbf{a}) \cdot x + (-\mathbf{3}) \cdot x^2 \\ 4x + ax^2 &= \mathbf{0} \cdot \mathbf{1} + \mathbf{4} \cdot x + \mathbf{a} \cdot x^2 \end{aligned}$$

we obtain that

$$P_{B' \leftarrow B} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 7 + a & 4 \\ 1 & -3 & a \end{pmatrix}.$$

Possible solution 1b:

only for part (i) We know that $B' = \{1, x, x^2\}$ is a basis of $P_2(\mathbb{R})$. Therefore $c_{B'}$ is an isomorphism and hence B is a basis of $P_2(\mathbb{R})$ if and only if $\{c_{B'}(p_1), c_{B'}(p_2), c_{B'}(p_3)\}$ is a basis of \mathbb{R}^3 . We have that

$$c_{B'}(p_1) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, c_{B'}(p_2) = \begin{pmatrix} 0 \\ 7 + a \\ -3 \end{pmatrix}, c_{B'}(p_3) = \begin{pmatrix} 0 \\ 4 \\ a \end{pmatrix}$$

We know that these vectors in \mathbb{R}^3 form a basis if and only if the matrix with these vectors as columns is invertible if and only if its determinant is non-zero.

$$\det \begin{pmatrix} 2 & 0 & 0 \\ 0 & 7 + a & 4 \\ 1 & -3 & a \end{pmatrix} = 2 \det \begin{pmatrix} 7 + a & 4 \\ -3 & a \end{pmatrix} = 2((7 + a)a + 12) = 2a^2 + 14a + 12$$

Using the pq -formula as in Solution 1a we obtain that this determinant is non-zero if and only if $a \neq -4$ and $a \neq -3$.

2. (i) Which of the following functions are linear? Justify your answer.

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x^2 \\ y \end{pmatrix}$$

$$g: \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 3x + z \\ x + y + z \end{pmatrix}$$

$$h: P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R}), \quad h(a_0 + a_1x + a_2x^2) = a_0x^2$$

- (ii) Choose one of the functions in (i) which is linear and determine a basis of its kernel, and a basis of its image.

Possible solution 2a: (i) The function f is not linear: We saw in the lecture that the function $\tilde{f}: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2$ is not linear. Furthermore since they are given by multiplication with matrices it follows that the following functions are linear: $\hat{f}: \mathbb{R} \rightarrow \mathbb{R}^2, x \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x$ and $\bar{f}: \mathbb{R}^2 \rightarrow \mathbb{R}, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Therefore if f were linear, then also $\tilde{f} = \bar{f} \circ f \circ \hat{f}$ would be linear, which we know it is not.

The function g is linear: It is given by multiplication with the matrix $\begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and therefore we know from the lecture that it is linear.

The function h is linear: We know that the function $\tilde{h}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by multiplication with the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is linear. We also know that $B = \{1, x, x^2\}$ is a

basis of $P_2(\mathbb{R})$. Therefore the function $h = c_B^{-1} \circ \tilde{h} \circ c_B$ is linear.

- (ii) We choose the function h . It is easy to see that $\text{Im}(h) = \text{span}(x^2)$ and since $x^2 \neq 0$ a basis of $\text{Im}(h)$ is given by $\{x^2\}$. By the rank-nullity theorem we therefore know that $\dim \ker(h) = 2$ (since $\dim P_2(\mathbb{R}) = 3$). It is easy to see that $h(x) = h(x^2) = 0$ and therefore $\{x, x^2\}$ gives a basis of $\ker(h)$.

Possible solution 2b: (i) The function f is not linear. We can compute that

$$f\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) = f\left(\begin{pmatrix} 4 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 16 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 8 \\ 0 \end{pmatrix} = f\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) + f\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right)$$

The function g is linear. We compute that

$$\begin{aligned} g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}\right) &= g\left(\begin{pmatrix} x+x' \\ y+y' \\ z+z' \end{pmatrix}\right) = \begin{pmatrix} 3(x+x') + z+z' \\ (x+x') + (y+y') + (z+z') \end{pmatrix} \\ &= \begin{pmatrix} 3x+z \\ x+y+z \end{pmatrix} + \begin{pmatrix} 3x'+z' \\ x'+y'+z' \end{pmatrix} = g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) + g\left(\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}\right) \end{aligned}$$

and

$$g\left(\lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = g\left(\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}\right) = \begin{pmatrix} 3(\lambda x) + (\lambda z) \\ \lambda x + \lambda y + \lambda z \end{pmatrix} = \lambda \begin{pmatrix} 3x + z \\ x + y + z \end{pmatrix} = \lambda g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)$$

Therefore, g is linear.

The function h is linear:

$$\begin{aligned} h((a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)) &= h((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2) = (a_0 + b_0)x^2 \\ &= a_0x^2 + b_0x^2 = h(a_0 + a_1x + a_2x^2) + h(b_0 + b_1x + b_2x^2) \end{aligned}$$

and

$$h(\lambda(a_0 + a_1x + a_2x^2)) = h((\lambda a_0) + (\lambda a_1)x + (\lambda a_2)x^2) = (\lambda a_0)x^2 = \lambda(a_0x^2) = \lambda h(a_0 + a_1x + a_2x^2)$$

(ii) We choose the function g . Since g is given by multiplication with the matrix $A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, the kernel of g is given by the null space of A . We perform Gaussian elimination to A to obtain a basis

$$\begin{pmatrix} 3 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{I-3II} \begin{pmatrix} 0 & -3 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

We see that a basis of the null space of g is given by $\left\{\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}\right\}$. Furthermore, the image of g is equal to the column space of A . The leading 1's are in the first and second column, therefore $\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ is a basis of $\text{Im}(g)$.

3. Let

$$A = \begin{pmatrix} -3 & -6 & 0 & 4 \\ -1 & -4 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ -3 & -9 & 0 & 5 \end{pmatrix}.$$

- (i) Show that the eigenvalues of A are 0 and -1 .
- (ii) For each eigenvalue, determine a basis of the corresponding eigenspace.
- (iii) Is A diagonalisable? Justify your answer.

Possible solution 3a: (i) We know that λ is an eigenvalue of A if and only if $\chi_A(\lambda) = \det(A - \lambda I_4) = 0$. We compute

$$\begin{aligned} \chi_A(\lambda) &= \det \begin{pmatrix} -3-\lambda & -6 & 0 & 4 \\ -1 & -4-\lambda & 0 & 2 \\ 0 & 0 & -1-\lambda & 0 \\ -3 & -9 & 0 & 5-\lambda \end{pmatrix} \\ &= (-1-\lambda) \det \begin{pmatrix} -3-\lambda & -6 & 4 \\ -1 & -4-\lambda & 2 \\ -3 & -9 & 5-\lambda \end{pmatrix} \\ &= (-1-\lambda) \left((-3-\lambda) \det \begin{pmatrix} -4-\lambda & 2 \\ -9 & 5-\lambda \end{pmatrix} - (-1) \det \begin{pmatrix} -6 & 4 \\ -9 & 5-\lambda \end{pmatrix} - 3 \det \begin{pmatrix} -6 & 4 \\ -4-\lambda & 2 \end{pmatrix} \right) \\ &= (-1-\lambda) ((-3-\lambda)((-4-\lambda)(5-\lambda) + 18) + (-6(5-\lambda) + 36) - 3(-12 + 4(4+\lambda))) \\ &= (-1-\lambda)(-\lambda^3 - 2\lambda^2 - \lambda) \\ &= (-1-\lambda)\lambda(-\lambda^2 - 2\lambda - 1) \\ &= (\lambda+1)^3 \lambda \end{aligned}$$

It follows that the eigenvalues of A are 0 and -1 .

- (ii) We compute bases of the corresponding eigenspaces. For $\lambda = 0$ we obtain $E(0, A) = N(A)$ which we compute using Gaussian elimination:

$$\begin{pmatrix} -3 & -6 & 0 & 4 \\ -1 & -4 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ -3 & -9 & 0 & 5 \end{pmatrix} \xrightarrow{I \leftrightarrow II} \begin{pmatrix} -1 & -4 & 0 & 2 \\ -3 & -6 & 0 & 4 \\ 0 & 0 & -1 & 0 \\ -3 & -9 & 0 & 5 \end{pmatrix} \xrightarrow{II-3I, IV-3I} \begin{pmatrix} -1 & -4 & 0 & 2 \\ 0 & 6 & 0 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix}$$

From this it is easy to see that a basis for the eigenspace is given by $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix} \right\}$.

For $\lambda = -1$ we obtain that $E(-1, A) = N(A + I_4)$ which we compute using Gaussian elimination:

$$\begin{pmatrix} -2 & -6 & 0 & 4 \\ -1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ -3 & -9 & 0 & 6 \end{pmatrix} \xrightarrow{I \leftrightarrow II} \begin{pmatrix} -1 & -3 & 0 & 2 \\ -2 & -6 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ -3 & -9 & 0 & 6 \end{pmatrix} \xrightarrow{II-2I, IV-3I} \begin{pmatrix} -1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that a basis for this eigenspace is given by $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$

- (iii) Yes, A is diagonalisable since for both eigenvalues the algebraic multiplicity coincides with the geometric multiplicity. For $\lambda = 0$ both are equal to 1 while for $\lambda = -1$ both are equal to 3.

Possible solution 3b: Note that in this solution we changed the order in which we solve the three parts of the exercise.

- (ii) We compute the eigenspaces of the eigenvalues 0 and -1 as in Solution 3a.
- (iii) Since by (ii) the geometric multiplicities of 0 and -1 are 1 and 3, respectively, we see that $1 + 3 = 4$, thus the sum of the geometric multiplicities is equal to the size of the matrix. Therefore the matrix is diagonalisable.
- (i) We know that eigenvectors corresponding to different eigenvalues are linearly independent. But according to (iii), \mathbb{R}^4 has a basis consisting of eigenvectors for A . Therefore, an eigenvector to a different eigenvalue cannot exist and therefore 0 and -1 are the only eigenvalues (we already checked in (ii) that they are indeed eigenvalues).

4. Let $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be a basis of $M_{2 \times 2}(\mathbb{R})$.

(i) Determine the coordinate vector of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ with respect to the basis B .

(ii) Let $f: M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ be the linear function given by $f(A) = \begin{pmatrix} 8 & 3 \\ 5 & 6 \end{pmatrix} A$. Determine the matrix $[f]_{B \leftarrow B}$ with respect to the basis B of $M_{2 \times 2}(\mathbb{R})$.

Possible solution 4a: (i) Since $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \mathbf{1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbf{3} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathbf{4} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, it follows

that the coordinate vector of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ with respect to the basis B is $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

(ii) We compute the image $f(b_i)$ of each of the basis vectors b_i in B and express them in the basis B :

$$f\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 8 & 0 \\ 5 & 0 \end{pmatrix} = \mathbf{8} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{0} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbf{5} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathbf{0} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 8 \\ 0 & 5 \end{pmatrix} = \mathbf{0} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{8} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbf{0} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathbf{5} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 3 & 0 \\ 6 & 0 \end{pmatrix} = \mathbf{3} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{0} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbf{6} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathbf{0} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 3 \\ 0 & 6 \end{pmatrix} = \mathbf{0} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{3} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbf{0} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathbf{6} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore,

$$[f]_{B' \leftarrow B} = \begin{pmatrix} 8 & 0 & 3 & 0 \\ 0 & 8 & 0 & 3 \\ 5 & 0 & 6 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}.$$

5. Let V be a vector space. Let $\{b_1, b_2, b_3\}$ be a basis of V .

- (i) Let $f: V \rightarrow \mathbb{R}^2$ be a function such that $f(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$. Prove that f is linear.
- (ii) Determine the matrix $[f]_{E \leftarrow B}$ of f with respect to the basis B of V and the standard basis E of \mathbb{R}^2 .

Possible solution 5a: (i) We know from the lecture that the function $c_B: V \rightarrow \mathbb{R}^3, \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 \mapsto \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$ is well-defined and linear. Furthermore we know that the

function $g: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by multiplication with $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is linear. Therefore the function $f = g \circ c_B$ is linear.

(ii) We have that

$$f(b_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f(b_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$f(b_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{Therefore } [f]_{E \leftarrow B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Possible solution 5b:

only for part (i) Since $\{b_1, b_2, b_3\}$ is a basis of V we know that every vector $v \in V$ has a unique expression as $v = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$. Let $v, v' \in V$. Write

$$v = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$$

$$v' = \mu_1 b_1 + \mu_2 b_2 + \mu_3 b_3$$

Then

$$f(v + v') = f((\lambda_1 + \mu_1)b_1 + (\lambda_2 + \mu_2)b_2 + (\lambda_3 + \mu_3)b_3) = \begin{pmatrix} \lambda_1 + \mu_1 \\ \lambda_2 + \mu_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = f(v) + f(v')$$

and

$$f(\lambda v) = f((\lambda \lambda_1)b_1 + (\lambda \lambda_2)b_2 + (\lambda \lambda_3)b_3) = \begin{pmatrix} \lambda \lambda_1 \\ \lambda \lambda_2 \end{pmatrix} = \lambda \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \lambda f(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) = \lambda f(v)$$

Therefore, f is linear.

6. Let $U = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix} \right\}.$

- (i) Give the definition of when a basis of an inner product space V is called orthonormal.
(ii) Find an orthonormal basis of U .

Possible solution 6a: (i) Let V be an inner product space with inner product $\langle -, - \rangle$. Then

a basis $\{b_1, \dots, b_n\}$ of V is called **orthonormal** if $\langle b_i, b_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$

- (ii) We compute an orthonormal basis of U using the Gram-Schmidt process starting

from the given basis of U . Call the given vectors $v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}.$

Then,

$$b'_1 = v_1, \quad \|b'_1\| = \sqrt{1+1+1+1} = 2, \quad b_1 = \frac{1}{\|b'_1\|} b'_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$b'_2 = v_2 - \frac{\langle v_2, b'_1 \rangle}{\langle b'_1, b'_1 \rangle} b'_1 = v_2 - \frac{-2}{4} b'_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ -2 \end{pmatrix}, \quad \|b'_2\| = \sqrt{\frac{9}{4} + \frac{9}{4} + \frac{1}{4} + \frac{1}{4}} = \sqrt{5}$$

$$b_2 = \frac{1}{\|b'_2\|} b'_2 = \frac{1}{2\sqrt{5}} \begin{pmatrix} 3 \\ 2 \\ 1 \\ -1 \end{pmatrix}$$

$$b'_3 = v_3 - \frac{\langle v_3, b'_1 \rangle}{\langle b'_1, b'_1 \rangle} b'_1 - \frac{\langle v_3, b'_2 \rangle}{\langle b'_2, b'_2 \rangle} b'_2 = v_3 - \frac{2}{4} b'_1 - \frac{3}{5} b'_2 = \begin{pmatrix} -\frac{2}{5} \\ \frac{2}{5} \\ \frac{6}{5} \\ -\frac{6}{5} \end{pmatrix}$$

$$\|b'_3\| = \sqrt{\frac{4}{25} + \frac{4}{25} + \frac{36}{25} + \frac{36}{25}} = \sqrt{\frac{80}{25}} = \frac{4\sqrt{5}}{5}$$

$$b_3 = \frac{1}{\|b'_3\|} b'_3 = \begin{pmatrix} -\frac{1}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} \\ \frac{3}{2\sqrt{5}} \\ -\frac{3}{2\sqrt{5}} \end{pmatrix}$$

7. Solve the following system of differential equations

$$y_1'(t) = 4y_1(t) + 2y_2(t)$$

$$y_2'(t) = -y_1(t) + y_2(t)$$

with the initial condition $y_1(0) = 2, y_2(0) = 3$.

Possible solution 7a: We write the system in matrix form:

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

As a next step we compute the eigenvalues for $\begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$ as the zeroes of its characteristic polynomial:

We obtain $\chi_A(\lambda) = \det \begin{pmatrix} 4-\lambda & 2 \\ -1 & 1-\lambda \end{pmatrix} = (4-\lambda)(1-\lambda) + 2 = \lambda^2 - 5\lambda + 6$. Using the

pq -formula we see that $\lambda_{1/2} = \frac{5}{2} \pm \sqrt{\frac{25}{4} - \frac{24}{4}} = \begin{cases} 3 \\ 2 \end{cases}$

The next step is to compute basis of the corresponding eigenspaces (which we know to be 1-dimensional as the geometric multiplicity is between 1 and the algebraic multiplicity which is also 1).

We have $A - 3I_2 = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$ and therefore a basis of $E(3, A)$ is given by $\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$.

We have that $A - 2I_2 = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$ and therefore a basis of $E(2, A)$ is given by $\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

With the substitution $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ we obtain the system $u_1'(t) = 3u_1(t), u_2'(t) = 2u_2(t)$ which has the solution $u_1 = c_1 e^{3t}, u_2 = c_2 e^{2t}$. Substituting back we obtain

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 2c_1 e^{3t} + c_2 e^{2t} \\ -c_1 e^{3t} - c_2 e^{2t} \end{pmatrix}$$

Taking into account the initial condition $y_1(0) = 2, y_2(0) = 3$ we obtain the additional condition that $\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2c_1 + c_2 \\ -c_1 - c_2 \end{pmatrix}$ which one sees to have the solution $c_1 = 5, c_2 = -8$.

Therefore a solution to the above system of differential equations with the above initial condition is given by

$$y_1(t) = 10e^{3t} - 8e^{2t}$$

$$y_2(t) = -5e^{3t} + 8e^{2t}$$

8. (i) On $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbb{R}, x + y = 2 \right\}$ define an addition \boxplus and a scalar multiplication \boxtimes via

$$\begin{pmatrix} x \\ y \end{pmatrix} \boxplus \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + x' - 1 \\ y + y' - 1 \end{pmatrix}$$

$$\lambda \boxtimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x - \lambda + 1 \\ \lambda y - \lambda + 1 \end{pmatrix}$$

(We checked in the lecture that this defines a vector space.)

Let W be the subspace of \mathbb{R}^2 given by $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x + y = 0 \right\}$.

Let $g: V \rightarrow W$ be the function defined by $g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix}$. Show that g is an isomorphism.

- (ii) What is $\dim V$? Justify your answer.

Possible solution 8a: (i) To show that g is an isomorphism we have to show that it is linear, injective, and surjective.

To show that it is linear we show that $g\left(\begin{pmatrix} x \\ y \end{pmatrix} \boxplus \begin{pmatrix} x' \\ y' \end{pmatrix}\right) = g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) + g\left(\begin{pmatrix} x' \\ y' \end{pmatrix}\right)$ and

$$g\left(\lambda \boxtimes \begin{pmatrix} x \\ y \end{pmatrix}\right) = \lambda g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

We have that

$$g\left(\begin{pmatrix} x \\ y \end{pmatrix} \boxplus \begin{pmatrix} x' \\ y' \end{pmatrix}\right) = g\left(\begin{pmatrix} x + x' - 1 \\ y + y' - 1 \end{pmatrix}\right) = \begin{pmatrix} x + x' - 2 \\ y + y' - 2 \end{pmatrix} = \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix} + \begin{pmatrix} x' - 1 \\ y' - 1 \end{pmatrix} = g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) + g\left(\begin{pmatrix} x' \\ y' \end{pmatrix}\right)$$

$$g\left(\lambda \boxtimes \begin{pmatrix} x \\ y \end{pmatrix}\right) = g\left(\begin{pmatrix} \lambda x - \lambda + 1 \\ \lambda y - \lambda + 1 \end{pmatrix}\right) = \begin{pmatrix} \lambda x - \lambda \\ \lambda y - \lambda \end{pmatrix} = \lambda \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix} = \lambda g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

Therefore g is linear.

To show that it is injective we show that $\ker(f) = \{0_V\}$. Assume that $g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

It follows from the definition of g that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We have shown in the lecture that this is the zero vector of V . Therefore f is injective.

To show that g is surjective let $\begin{pmatrix} x \\ y \end{pmatrix} \in W$. It is easy to see that $g\left(\begin{pmatrix} x + 1 \\ y + 1 \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}$.

Therefore g is surjective.

- (ii) We know that W is one-dimensional since it is given as the null space of a rank 1 matrix with 2 columns. By a result in the lecture we know that isomorphic spaces have the same dimension. Therefore $\dim V = 1$.

Possible solution 8b: (ii) The dimension of a vector space V is defined to be the number of basis vectors in a basis for V . We claim that $\left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$ is a basis for V . Note that

the zero vector in V is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, therefore $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ is not the zero vector, and this set is linearly independent. We show that it is also spanning. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in V$. We know that $y = 2 - x$. Therefore $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2 - x \end{pmatrix} = (1 - x) \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Therefore the set is also spanning and hence a basis. It follows that $\dim V = 1$.

- (i) We show that g is linear in the same way as in Solution 8a. Since we know that $\dim W = 1 = \dim V$ (see Solution 8a and part (ii) of Solution 8b) it suffices to prove that g is injective to show that g is an isomorphism. Assume that $g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = g\left(\begin{pmatrix} x' \\ y' \end{pmatrix}\right)$. Then $\begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix} = \begin{pmatrix} x' - 1 \\ y' - 1 \end{pmatrix}$ and therefore $x = x'$ and $y = y'$. Thus, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ and it follows that g is injective (and therefore an isomorphism).