TENTAMEN - LINEAR ALGEBRA II 2020/03/16

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English version

Time: 8.00-13.00. No aids allowed except a pen. All solutions should be accompanied with justifications.
Each of the following exercises is worth 5 points, i.e. the total score of the tenta is 40 points. If you achieve 18, 25, or 32 points, respectively, you will receive grade 3,4, or 5.
Up to 4 bonus points from the dugga on 2020/02/21 can be used for this tentamen.

1. (i) Which of the following subsets are subspaces of the given vector spaces? Justify your answer.

- $U_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x + 3y + z = 0, \quad y - 2z = 0 \right\}$,
- $U_2 = \{p \in P_{\leq 4}(\mathbb{R}) \,|\, p(x) = p(-x)\}$,
- $U_3 = \left\{ A \in M_{2\times 2}(\mathbb{R}) \,\middle|\, A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

(ii) For each subspace in (i) determine a basis and give its dimension.

Possible solution 1a: (i)

- We see that $U_1 = N(A)$, the null space of the matrix $A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -2 \end{pmatrix}$. We know from the lecture that the null space of every matrix $A \in M_{m\times n}(\mathbb{R})$ is a subspace of $\mathbb{R}^n$, therefore U_1 is a subspace of $\mathbb{R}^3$.
- We have to check the subspace criteria, i.e. that U_2 is not empty, closed under addition, and scalar multiplication. For the former, we see that the constant zero polynomial 0 is in U_2 as $0(x) = 0(-x)$ for all $x \in \mathbb{R}$. To check that U_2 is closed under addition let $p, q \in U_2$. Then $p(x) = p(-x)$ and $q(x) = q(-x)$ for all $x \in \mathbb{R}$. Therefore, $(p + q)(x) = p(x) + q(x) = p(-x) + q(-x) = (p + q)(-x)$ for all $x \in \mathbb{R}$. Thus, $p + q \in U_2$. To check that U_2 is closed under scalar multiplication, let $p \in U_2$ and $\lambda \in \mathbb{R}$. Then $p(x) = p(-x)$ for all $x \in \mathbb{R}$ and therefore $(\lambda p)(x) = \lambda(p(x)) = \lambda(p(-x)) = (\lambda p)(-x)$ for all $x \in \mathbb{R}$ and therefore $\lambda p \in U_2$.
- The subset U_3 is not a subspace of $M_{2\times 2}(\mathbb{R})$ as the zero vector of $M_{2\times 2}(\mathbb{R})$, that is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ doesn't satisfy $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(ii)

- We can determine the dimension of U_1 by means of the rank-nullity theorem. This says that $\dim N(A) = n - \mathrm{rk}(A)$ where n is the number of columns of A. In this case we get that $\dim U_1 = \dim N(A) = 3 - 2 = 1$. Therefore, every non-zero vector in U_1 forms a basis of U_1, an example of a basis is given by $\left\{ \begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix} \right\}$.

- To find a basis for U_2 we write out the condition $p(x) = p(-x)$ for an arbitrary polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$. It means that

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 = a_0 - a_1x + a_2x^2 - a_3x^3 + a_4x^4.$$

  Comparing coefficients we see that $a_1 = 0$ and $a_3 = 0$ are the only conditions. Therefore U_2 is spanned by $1, x^2, x^4$ which are also linearly independent and thus form a basis of U_2. Since U_2 has a basis with three elements, it is three-dimensional.

Possible solution 1b: (i)

- We use the subspace criteria to show that U_1 is a subspace by showing it is not empty, closed under addition and scalar multiplication. For the former note that $\begin{pmatrix}0\\0\\0\end{pmatrix} \in U_1$. Let $\begin{pmatrix}x\\y\\z\end{pmatrix}, \begin{pmatrix}x'\\y'\\z'\end{pmatrix} \in U_1$. This means that $x + 3y + z = 0, y - 2z = 0$ and $x' + 3y' + z' = 0, y' - 2z' = 0$. We want to check that $\begin{pmatrix}x+x'\\y+y'\\z+z'\end{pmatrix}$ is also in U_1. We check the conditions:

$$(x + x') + 3(y + y') + (z + z') = (x + 3y + z) + (x' + 3y' + z') = 0$$

  and

$$(y + y') - 2(z + z') = (y - 2z) + (y' - 2z') = 0.$$

  We can conclude that $\begin{pmatrix}x+x'\\y+y'\\z+z'\end{pmatrix} \in U_1$. Similarly if $\lambda \in \mathbb{R}$ and $\begin{pmatrix}x\\y\\z\end{pmatrix} \in U_1$ then $\lambda\begin{pmatrix}x\\y\\z\end{pmatrix} \in U_1$ since $(\lambda x) + 3(\lambda y) + \lambda z = \lambda(x + 3y + z) = 0$ and $\lambda y - 2(\lambda z) = \lambda(y - 2z) = 0$.
- Define a map $f\colon P_{\leq 4}(\mathbb{R}) \to P_{\leq 4}(\mathbb{R})$ via $f(p) = p(x) - p(-x)$. We check that this map is linear: $f(p + q) = (p + q)(x) - (p + q)(-x) = (p(x) - p(-x)) + (q(x) - q(-x)) = f(p) + f(q)$ and $f(\lambda p) = (\lambda p)(x) - (\lambda p)(-x) = \lambda(p(x) - p(-x)) = \lambda f(p)$. We conclude that U_2 is a subspace since it is equal to the kernel of f, which we know is a subspace using results of the lecture.
- The subset U_3 is not a subspace since for example $A = \begin{pmatrix}1&0\\0&1\end{pmatrix} \in U_3$, but $2A = \begin{pmatrix}2&0\\0&2\end{pmatrix}$ is not in U_3 since $(2A)^2 = 4A \neq \begin{pmatrix}1&0\\0&1\end{pmatrix}$ and therefore U_3 is not closed under scalar multiplication.

(ii)

- We solve the linear system of equations $x+3y+z = 0$ and $y-2z = 0$. We see immediately that there is one free parameter $z = t$ and that x and y are bound variables given as $y = 2t$ and $x = -7t$. Therefore, $U_1 = \mathrm{span}(\begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix})$ and since $\begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix}$ is non-zero, it is linearly independent and thus $\left\{ \begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix} \right\}$ is a basis of U_1. Since the basis only has one basis vector, $\dim U_1 = 1$.

- To compute the dimension of U_2, we use the dimension formula for linear maps for the linear map f defined in (i). This says that

$$\dim P_{\leq 4}(\mathbb{R}) = \dim \ker(f) + \dim \mathrm{Im}(f).$$

Therefore, $\dim U_2 = \dim P_{\leq 4}(\mathbb{R}) - \dim \mathrm{Im}(f)$. To compute the dimension of $\mathrm{Im}(f)$ we compute $f(p)$ for some $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$. We see that $f(p) = 2a_1 x + 2a_3 x^3$ and therefore $\mathrm{Im}(f) = \mathrm{span}(x, x^3)$. It follows that $\dim \mathrm{Im}(f) = 2$ and thus $\dim U_2 = 5 - 2 = 3$. It therefore suffices to find three linearly independent vectors in U_2. It is easy to see that $1, x^2, x^4$ are such vectors and therefore $\{1, x^2, x^4\}$ is a basis of U_2.

2. Let $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$ and $C = \left\{ \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$.

(i) Prove that B and C are bases of $\mathbb{R}^3$.
(ii) Compute the base change matrix $P_{B\leftarrow C}$ from C to B (*the notation of the course book is* $P_{C\to B}$).
(iii) Let $v \in \mathbb{R}^3$ be the vector which satisfies $[v]_C = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Determine $[v]_B$.

Possible solution 2a: (i) Since $\dim \mathbb{R}^3 = 3$ it suffices to show that B and C are linearly independent. To show that B is linearly independent we have to show that the linear system of equations

$$\lambda_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has only the trivial solution. We solve the system using Gaussian elimination:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 2 \end{pmatrix} \overset{II-I,III+I}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

This has only the trivial solution and therefore B is a basis of $\mathbb{R}^3$. Similarly, to show that C is linearly independent we have to show that the linear system of equations

$$\lambda_1 \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has only the trivial solution. We solve the system using Gaussian elimination:

$$\begin{pmatrix} -3 & 1 & -1 \\ 1 & 1 & 2 \\ -1 & 3 & 1 \end{pmatrix} \overset{I\leftrightarrow II}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & 2 \\ -3 & 1 & -1 \\ -1 & 3 & 1 \end{pmatrix} \overset{II+3\cdot I,III+I}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{pmatrix} \overset{III-II}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & -2 \end{pmatrix}$$

At this stage we can see that this has only the trivial solution and therefore C is a basis of $\mathbb{R}^3$.

(ii) The base change matrix $P_{B\leftarrow C}$ is the matrix whose columns are the coordinate vectors of the vectors in C with respect to the basis B. To compute it, we thus have to solve the following three linear systems of equations:

$$\lambda_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}$$

$$\mu_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \mu_2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu_3 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\nu_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \nu_2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \nu_3 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Since the left hand sides coincide we can solve it using the following Gaussian elimination:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & -3 & 1 & -1 \\ 1 & 2 & 1 & 1 & 1 & 2 \\ -1 & -1 & 2 & -1 & 3 & 1 \end{array}\right) \overset{II-I,III+I}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & -3 & 1 & -1 \\ 0 & 1 & 2 & 4 & 0 & 3 \\ 0 & 0 & 1 & -4 & 4 & 0 \end{array}\right)$$

$$\overset{II-2\cdot III,I+III}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -7 & 5 & -1 \\ 0 & 1 & 0 & 12 & -8 & 3 \\ 0 & 0 & 1 & -4 & 4 & 0 \end{array}\right) \overset{I-II}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -19 & 13 & -4 \\ 0 & 1 & 0 & 12 & -8 & 3 \\ 0 & 0 & 1 & -4 & 4 & 0 \end{array}\right)$$

We have $\left[\begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}\right]_B = \begin{pmatrix} -19 \\ 12 \\ -4 \end{pmatrix}$, $\left[\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}\right]_B = \begin{pmatrix} 13 \\ -8 \\ 4 \end{pmatrix}$, $\left[\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right]_B = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$ and therefore we obtain

$$P_{B\leftarrow C} = \begin{pmatrix} -19 & 13 & -4 \\ 12 & -8 & 3 \\ -4 & 4 & 0 \end{pmatrix}.$$

(iii) We use the formula $P_{B\leftarrow C}[v]_C = [v]_B$ to obtain that

$$[v]_B = \begin{pmatrix} -19 & 13 & -4 \\ 12 & -8 & 3 \\ -4 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 7 \\ 0 \end{pmatrix}$$

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Possible solution 2b: (i) Since $\dim \mathbb{R}^3 = 3$, it suffices to show that B and C are linearly independent. For showing that B is linearly independent we have to show that

$$\lambda_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

only has the trivial solution. From Linear Algebra I we know that we can check this by computing the determinant of the corresponding matrix.

$$\det(\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 2 \end{pmatrix}) = 1 \cdot \det(\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}) - 1 \cdot \det(\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}) - 1 \det(\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix})$$
$$= 1 \cdot (4+1) - 1 \cdot (2-1) - 1 \cdot (1+2) = 1.$$

Since the determinant is not equal to zero, it follows that B is a basis of $\mathbb{R}^3$. Similarly for C we compute that

$$\det(\begin{pmatrix} -3 & 1 & -1 \\ 1 & 1 & 2 \\ -1 & 3 & 1 \end{pmatrix}) = -3 \det(\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}) - 1 \det(\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}) - 1 \det(\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix})$$
$$= -3 \cdot (1-6) - 1 \cdot (1+2) - 1 \cdot (3+1) = 8$$

(ii) We use the formula $P_{B\leftarrow C} = P_{B\leftarrow E} P_{E\leftarrow C} = (P_{E\leftarrow B})^{-1} P_{E\leftarrow C}$, where E is the standard basis of $\mathbb{R}^3$. The matrix $P_{E\leftarrow C}$ is the matrix whose columns are the basis vectors of C, i.e. $P_{E\leftarrow C} = \begin{pmatrix} -3 & 1 & -1 \\ 1 & 1 & 2 \\ -1 & 3 & 1 \end{pmatrix}$. We need to compute the inverse of the matrix $P_{E\leftarrow B}$ using Gaussian elimination:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \overset{II-I, III+I}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$
$$\overset{I+III, II-2\cdot III}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -3 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \overset{I-II}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -1 & 3 \\ 0 & 1 & 0 & -3 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

Therefore,

$$P_{B\leftarrow C} = \begin{pmatrix} 5 & -1 & 3 \\ -3 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 & 1 & -1 \\ 1 & 1 & 2 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} -19 & 13 & -4 \\ 12 & -8 & 3 \\ -4 & 4 & 0 \end{pmatrix}$$

(iii) Let v be the vector with $[v]_C = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. This means that

$$v = 1 \cdot \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix}$$

To determine the coordinate vector of v with respect to the basis B, we need to write v as a linear combination of the vectors in B, i.e. find $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such

that
$$\begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$
We solve this system using Gaussian elimination:
$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 1 & 2 & 1 & 4 \\ -1 & -1 & 2 & 3 \end{array}\right) \overset{II-I, III+I}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right)$$
We apply backwards substitution and obtain $\lambda_3 = 0$, $\lambda_2 = 7$, and $\lambda_1 = -10$. In other words,
$$[v]_C = \begin{pmatrix} -10 \\ 7 \\ 0 \end{pmatrix}$$

3. Let $P_{\leq n}(\mathbb{R})$ be the space of all polynomials of degree at most n.

(i) Determine whether or not the polynomials

$$p_1(x) = 1 - x^2,$$
$$p_2(x) = 2 + 5x + x^2,$$
$$p_3(x) = -4x + 3x^2$$

are linearly independent in $P_{\leq 2}(\mathbb{R})$.

(ii) Show that the function $f\colon P_{\leq 2}(\mathbb{R}) \to P_{\leq 1}(\mathbb{R})$ given by $f(a + bx + cx^2) = b + cx$ is linear.

Possible solution 3a: (i) We have to determine the solutions to

$$\lambda_1(1 - x^2) + \lambda_2(2 + 5x + x^2) + \lambda_3(-4x + 3x^2) = 0$$

Comparing coefficients this yields the system of equations

$$\lambda_1 + 2\lambda_2 = 0$$
$$5\lambda_2 - 4\lambda_3 = 0$$
$$-\lambda_1 + \lambda_2 + 3\lambda_3 = 0$$

We solve this system using Gaussian elimination:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 5 & -4 \\ -1 & 1 & 3 \end{pmatrix} \overset{III+I}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 5 & -4 \\ 0 & 3 & 3 \end{pmatrix} \overset{III-\frac{3}{5}\cdot II}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 5 & -4 \\ 0 & 0 & \frac{27}{5} \end{pmatrix}$$

Thus, the system has only the trivial solution and therefore, p_1, p_2, and p_3 are linearly independent.

(ii) To show that f is linear, we have to show that $f(p + q) = f(p) + f(q)$ and $f(\lambda p) = \lambda f(p)$ for all $p, q \in P_{\leq 2}(\mathbb{R})$ and all $\lambda \in \mathbb{R}$. To show the former let $p(x) = a_1 + b_1x + c_1x^2$ and $q(x) = a_2 + b_2x + c_2x^2$ be in $P_{\leq 2}(\mathbb{R})$. Then

$$\begin{aligned} f(p + q) &= f((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2) = (b_1 + b_2) + (c_1 + c_2)x \\ &= (b_1 + c_1x) + (b_2 + c_2x) = f(p) + f(q) \end{aligned}$$

and

$$f(\lambda p) = f((\lambda a_1) + (\lambda b_1)x + (\lambda c_1)x^2) = \lambda b_1 + (\lambda c_1)x = \lambda(b_1 + c_1x) = \lambda f(p).$$

Thus, f is linear.

Possible solution 3b: (i) Let $B = \{1, x, x^2\}$ be a basis of $P_{\leq 2}(\mathbb{R})$. Since

$$c_B\colon P_{\leq 2}(\mathbb{R}) \to \mathbb{R}^3, p \mapsto [p]_B$$

is injective, we know that $\{p_1, p_2, p_3\}$ is linearly independent if and only if

$$\{c_B(p_1),\, c_B(p_2),\, c_B(p_3)\}$$

are linearly independent. We see that

$$c_B(p_1) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \; c_B(p_2) = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \; c_B(p_3) = \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix}.$$

To show whether these are linearly independent, we have to determine whether

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

only has the trivial solution. To determine this, we know that we can check whether the determinant of the corresponding matrix is non-zero:

$$\det(\begin{pmatrix} 1 & 2 & 0 \\ 0 & 5 & -4 \\ -1 & 1 & 3 \end{pmatrix}) = 1 \cdot \det(\begin{pmatrix} 5 & -4 \\ 1 & 3 \end{pmatrix}) + (-1) \cdot \det(\begin{pmatrix} 2 & 0 \\ 5 & -4 \end{pmatrix}) = 19 + 8 = 27.$$

Thus, p_1, p_2, p_3 are linearly independent.

(ii) Recall c_B from part (i) is an invertible linear map and the analogous map $c_{B'} : P_{\leq 1}(\mathbb{R}) \to \mathbb{R}^2, p \mapsto [p]_{B'}$ where $B' = \{1, x\}$ is also invertible and linear. Since the composition of linear maps is linear, it suffices to show that $g = c_{B'} f c_B^{-1}$ is linear (since then $f = c_{B'}^{-1} g c_B$ is linear). We compute that

$$g(\begin{pmatrix} a \\ b \\ c \end{pmatrix}) = c_{B'} f c_B^{-1}(\begin{pmatrix} a \\ b \\ c \end{pmatrix}) = c'_B f(a + bx + cx^2)$$

$$= c'_B(b + cx) = \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Thus, $g \colon \mathbb{R}^3 \to \mathbb{R}^2$ is given by multiplication with a matrix and therefore linear. It follows that f is linear.

4. (i) Compute the eigenvalues of $A = \begin{pmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$.

(ii) For each eigenvalue, determine a basis of the corresponding eigenspace.

(iii) What are the eigenvalues and corresponding eigenspaces of A^3? Justify your answer.

Possible solution 4a: (i) The eigenvalues of A are the zeroes of the characteristic polynomial $\chi_A(\lambda)$ of A.

$$\begin{aligned}
-\chi_A(\lambda) &= \det(A - \lambda I_3) = \det(\begin{pmatrix} 3-\lambda & -1 & -2 \\ -1 & 3-\lambda & -2 \\ -2 & -2 & 4-\lambda \end{pmatrix}) \\
&= (3-\lambda)\det(\begin{pmatrix} 3-\lambda & -2 \\ -2 & 4-\lambda \end{pmatrix}) + 1\det(\begin{pmatrix} -1 & -2 \\ -2 & 4-\lambda \end{pmatrix}) - 2\det(\begin{pmatrix} -1 & -2 \\ 3-\lambda & -2 \end{pmatrix}) \\
&= (3-\lambda)((3-\lambda)(4-\lambda) - 4) + (-(4-\lambda) - 4) - 2(2 + 2(3-\lambda)) \\
&= -\lambda(\lambda - 4)(\lambda - 6)
\end{aligned}$$

Thus the eigenvalues of A are 0, 4, and 6.

(ii) The eigenspaces of A are the null spaces of $A - \lambda I$ for the eigenvalues λ. We compute them using Gaussian elimination. For $\lambda = 0$:

$$\begin{pmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{pmatrix} \overset{I \leftrightarrow II}{\rightsquigarrow} \begin{pmatrix} -1 & -3 & -2 \\ 3 & -1 & -2 \\ -2 & -2 & 4 \end{pmatrix} \overset{II+3\cdot I, III-2\cdot I}{\rightsquigarrow} \begin{pmatrix} -1 & -3 & -2 \\ 0 & 8 & -8 \\ 0 & -8 & 8 \end{pmatrix} \overset{III+II}{\rightsquigarrow} \begin{pmatrix} 1 & -3 & -2 \\ 0 & 8 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

Using backwards substitution, we obtain that $\lambda_1 = \lambda_2 = \lambda_3$. Thus, a basis of $E(0, A)$ is given by $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

For $\lambda = 4$:

$$\begin{pmatrix} -1 & -1 & -2 \\ -1 & -1 & -2 \\ -2 & -2 & 0 \end{pmatrix} \overset{II-I, III-2\cdot I}{\rightsquigarrow} \begin{pmatrix} -1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Using backwards substitution, we obtain $\lambda_3 = 0$ and $\lambda_1 = -\lambda_2$. Thus, a basis of $E(4, A)$ is given by $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$.

For $\lambda = 6$:

$$\begin{pmatrix} -3 & -1 & -2 \\ -1 & -3 & -2 \\ -2 & -2 & -2 \end{pmatrix} \overset{I \leftrightarrow II}{\rightsquigarrow} \begin{pmatrix} -1 & -3 & -2 \\ -3 & -1 & -2 \\ -2 & -2 & -2 \end{pmatrix} \overset{II-3\cdot I, III-2\cdot I}{\rightsquigarrow} \begin{pmatrix} -1 & -3 & -2 \\ 0 & 8 & 4 \\ 0 & 4 & 2 \end{pmatrix} \overset{III-\frac{1}{2}II}{\rightsquigarrow} \begin{pmatrix} -1 & -3 & -2 \\ 0 & 8 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Using backwards substitution, we obtain $\lambda_2 = -\frac{1}{2}\lambda_3$, $\lambda_1 = -\frac{1}{2}\lambda_3$. Therefore, $\left\{\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}\right\}$ is a basis of $E(6, A)$.

(iii) An eigenvector $v \in \mathbb{R}^3$ corresponding to an eigenvalue $\lambda \in \mathbb{R}$ is a vector $v \neq 0$, such that $Av = \lambda v$. For such a vector, $A^3 v = A(A(Av)) = \lambda^3 v$. Therefore, 0, $4^3 = 64$ and $6^3 = 216$ are eigenvalues of A. Since a 3×3-matrix can have at most three eigenvalues, these are all the eigenvalues of A. The previous calculation also shows that the eigenspaces for A and for A^3 coincide.

Possible solution 4b: (i) We see that the sum of the first three rows of the matrix is 0. This means that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for A. Since the characteristic polynomial does not depend on the choice of basis, we let $B = \left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$. If $f\colon \mathbb{R}^3 \to \mathbb{R}^3, v \mapsto Av$, then $[f]_{E\leftarrow E} = A$ and therefore $[f]_{B\leftarrow B} = P_{B\leftarrow E} A P_{E\leftarrow B} = P^{-1}_{E\leftarrow B} A P_{E\leftarrow B}$. We know that $P_{E\leftarrow B} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. We compute $P^{-1}_{E\leftarrow B}$ using Gaussian elimination:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \overset{III\leftrightarrow I}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$\overset{II-I, III-I}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right) \overset{II\leftrightarrow III}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right)$$

It follows that

$$[f]_{B\leftarrow B} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -2 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

Thus, the characteristic polynomial of A is equal to $\lambda((\lambda-5)^2-1) = \lambda(\lambda-4)(\lambda-6)$ and the eigenvalues are 0, 4, and 6.

(ii) The eigenvectors of $A' = [f]_{B\leftarrow B}$ are the coordinate vectors of the eigenvectors of A with respect to the basis B. We know that $N(A') = \mathrm{span}(e_1)$ and therefore $N(A) = \mathrm{span}(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix})$ and a basis of $E(0, A)$ is given by $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$. Gaussian elimination for $\lambda = 4$ yields

$$\begin{pmatrix} -4 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \overset{III-II}{\rightsquigarrow} \begin{pmatrix} -4 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and backwards substitution yields $\lambda_1 = 0$ and $\lambda_2 = -\lambda_3$. Therefore, $E(4, A') = \operatorname{span}(\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix})$ and $E(4, A) = \operatorname{span}(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix})$ with basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. Gaussian elimination for $\lambda = 6$ yields

$$\begin{pmatrix} -6 & -2 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \overset{III+II}{\rightsquigarrow} \begin{pmatrix} -6 & -2 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Backwards substitution yields $\lambda_2 = \lambda_3$ and $\lambda_1 = -\frac{2}{3}\lambda_2$. Thus, $E(6, A') = \operatorname{span}(\begin{pmatrix} -\frac{2}{3} \\ 1 \\ 1 \end{pmatrix}$ and $E(6, A) = \operatorname{span}(\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix})$ with basis $\left\{ \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} \right\}$.

5. Consider the basis $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ of $M_{2\times 2}(\mathbb{R})$ and the basis $C = \{1, x\}$ of $P_{\leq 1}(\mathbb{R})$.

(i) Determine the coordinate vector of $\begin{pmatrix} 2 & 4 \\ 1 & 8 \end{pmatrix}$ with respect to the basis B and the coordinate vector of $2 - 7x$ with respect to the basis C.

(ii) Let $f \colon M_{2\times 2}(\mathbb{R}) \to P_{\leq 1}(\mathbb{R})$ be the linear map given by

$$f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (a + d) + (b + c)x.$$

Determine the matrix of f with respect to the bases B of $M_{2\times 2}(\mathbb{R})$ and C of $P_{\leq 1}(\mathbb{R})$.

(iii) Give a basis of the kernel $\ker(f)$.

Possible solution 5a: (i) Since

$$\begin{pmatrix} 2 & 4 \\ 1 & 8 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 8 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

it follows that $\left[\begin{pmatrix} 2 & 4 \\ 1 & 8 \end{pmatrix} \right]_B = \begin{pmatrix} 2 \\ 4 \\ 1 \\ 8 \end{pmatrix}$. Similarly since $2 - 7x = 2 \cdot 1 + (-7) \cdot x$ it follows that $[2 - 7x]_C = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$.

(ii) The columns of the matrix f are the coordinate vectors of the images of the basis vectors in B with respect to the basis C. Thus, we compute

$$f(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = 1 = 1 \cdot 1 + 0 \cdot x$$

$$f(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = x = 0 \cdot 1 + 1 \cdot x$$

$$f(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = x = 0 \cdot 1 + 1 \cdot x$$

$$f(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = 1 = 1 \cdot 1 + 0 \cdot x$$

It follows that

$$[f]_{C \leftarrow B} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

(iii) If we let $A = [f]_{C \leftarrow B}$, then the kernel of f consists of those matrices v such that $[v]_B \in N(A)$, the null space of A. We see that $N(A) = \mathrm{span}(\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix})$

and therefore $\ker(f) = \operatorname{span}(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix})$ and a basis of $\ker(f)$ is given by $\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

Possible solution 5b: (i) As in Possible solution 5a.

(ii) The matrix of f is the matrix $[f]_{C\leftarrow B}$ which satisfies $[f(v)]_C = [f]_{C\leftarrow B}[v]_B$. We have that $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]_B = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ and $[f(v)]_C = [(a+d)+(b+c)x]_C = \begin{pmatrix} a+d \\ b+c \end{pmatrix}$.

Therefore, we want to find the matrix $[f]_{C\leftarrow B}$ such that

$$\begin{pmatrix} a+d \\ b+c \end{pmatrix} = [f]_{C\leftarrow B} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

We see that this matrix is $[f]_{C\leftarrow B} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$.

(iii) The kernel of f is defined as

$$\ker(f) = \{v \in M_{2\times 2}(\mathbb{R}) \mid f(v) = 0\}$$

We have that $f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (a+d)+(b+c)x$. In order for that to be equal to the zero polynomial, we need that $a+d=0$ and $b+c=0$. Therefore, a basis of $\ker(f)$ is given by $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$.

6. Let V be an inner product space and let $U \subseteq V$ be a subspace.

(i) Give the definition of the orthogonal complement $U^\perp$ of U in V.

(ii) For $U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x + 2y - z = 0 \right\}$ determine orthonormal bases of U and of $U^\perp$.

Possible solution 6a: (i) The orthogonal complement $U^\perp$ is defined as

$$U^\perp = \{v \in V \mid \langle u, v\rangle = 0 \text{ for all } u \in U\}.$$

(ii) We start by determining bases of U and of $U^\perp$. The condition $x + 2y - z = 0$ can be rewritten as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0.$$

Thus, a basis for the orthogonal complement of U is given by $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$. Solving $x + 2y - z = 0$ we see that a basis of U is given by $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$. We now apply the Gram–Schmidt orthonormalisation procedure to obtain orthonormal basis. For $U^\perp$, we only have to normalize as there is only one vector. Since

$$\left\| \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$$

we obtain that $B' = \left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$ is an orthonormal basis of $U^\perp$. Applying the Gram–Schmidt procedure to the basis of U gives:

$$b_1' = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$b_2' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{2}{5} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ 1 \end{pmatrix}$$

These two basis vectors form an orthogonal basis of U. To determine an orthonormal basis, we need to normalise them. We have computed that $\|b_1'\| = \sqrt{5}$ and $\|b_2'\| = \sqrt{(\frac{1}{5})^2 + (\frac{2}{5})^2 + 1} = \sqrt{\frac{6}{5}}$. Therefore, an orthonormal basis of U is given by $\left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \frac{\sqrt{5}}{\sqrt{6}} \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ 1 \end{pmatrix} \right\}$.

Possible solution 6b: (i) An alternative way to define the orthogonal complement of a subspace U is to use the projection onto the subspace U. Let $\{u_1, \dots, u_m\}$ be an orthogonal basis of U (which exists by the Gram–Schmidt orthonormalisation procedure). Then the orthogonal complement $U^\perp$ is the kernel of the orthogonal projection proj_U defined by:

$$\mathrm{proj}_U(v) = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 + \cdots + \frac{\langle v, u_m \rangle}{\langle u_m, u_m \rangle} u_m$$

(ii) As in Possible solution 6a we see that $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$ is an orthogonal basis of $U^\perp$ and we can normalise it to obtain the orthonormal basis $\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$. Secondly, we observe that the vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is in U. Note that a second basis vector of an orthonormal basis of U has to be both orthogonal to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ as well as $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. From Linear Algebra I, we know that such a vector is given by the cross product of the two vectors:

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$$

We finish by normalising these two vectors to obtain the orthonormal basis

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

of U.

7. (i) What kind of curve is $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, 7x^2 + 4xy + y^2 = 1 \right\}$?

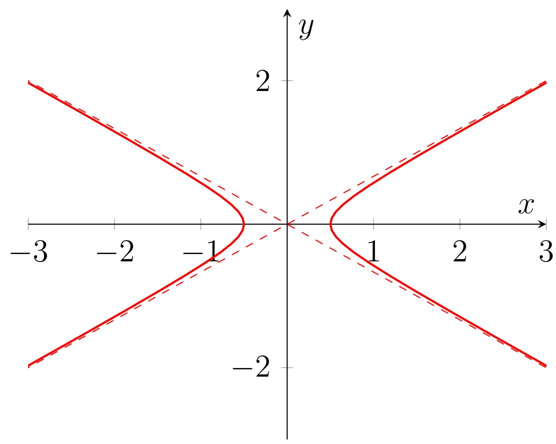
(ii) Sketch the set $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, 4x^2 - 9y^2 = 1 \right\}$ in the xy-plane.

Possible solution 7: (i) We first write $7x^2 + 4xy + y^2 = 1$ in matrix form:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 7 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1$$

To show what shape the curve is, we compute the eigenvalues of the matrix $\begin{pmatrix} 7 & 2 \\ 2 & 1 \end{pmatrix}$. Its characterisitic polynomial is $(\lambda - 7)(\lambda - 1) - 4 = \lambda^2 - 8\lambda + 3$. Its eigenvalues are $4 \pm \sqrt{13}$, which are both positive ($\sqrt{13} < \sqrt{16} = 4$). Therefore, there is an isometry P such that the transformed equation satisfies $(4 + \sqrt{13})(x')^2 + (4 - \sqrt{13})(y')^2 = 1$. This is an ellipse and since isometries don't change the shape of curves, it follows that the original curve is an ellipse as well.

(ii) The set is a hyperbola with asymptotes $y = \pm\frac{2}{3}x$ and intersections with the axes at $(\pm\frac{1}{2}, 0)$.



8. Let V and W be vector spaces.

(i) Let $v_1 \neq v_2 \in V$. Let $f\colon V \to W$ be a linear map such that $f(v_1) = f(v_2) \neq 0_W$. Show that v_1 and v_2 are linearly independent.

(ii) Let $f, g\colon V \to W$ be linear maps and let $u, v \in V$ such that $f(u) = g(u) \neq 0_W$ and $f(v) = -g(v) \neq 0_W$. Show that f and g are linearly independent in the space of all functions $V \to W$.

Possible solution 8a: (i) We have to show that the equation $\lambda_1 v_1 + \lambda_2 v_2 = 0_V$ has only the trivial solution for λ_1 and λ_2. We apply f to the above equation and using linearity of f we obtain that

$$0_W = f(0_V) = f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2) = (\lambda_1 + \lambda_2) f(v_1)$$

By assumption $f(v_1) \neq 0_W$ and therefore $\lambda_1 + \lambda_2 = 0$. Therefore, the original equation reads as $\lambda_1 v_1 - \lambda_1 v_2 = 0$. If $\lambda_1 \neq 0$, dividing by λ_1 yields $v_1 = v_2$, which is a contradiction. Therefore we must have $\lambda_1 = 0$ and then $\lambda_2 v_2 = 0$ implies $\lambda_2 = 0$ as v_2 can't be the zero-vector since $f(v_2) \neq 0_W$. It follows that v_1 and v_2 are linearly independent.

(ii) To show that f and g are linearly independent, we have to show that the equation $\lambda_1 f + \lambda_2 g = 0$ only has the trivial solution $\lambda_1 = \lambda_2 = 0$. The equation $\lambda_1 f + \lambda_2 g = 0$ means that $\lambda_1 f + \lambda_2 g$ is the constant zero function, i.e. its value is zero for every vector. Therefore we obtain the equations

$$\lambda_1 f(u) + \lambda_2 g(u) = 0_W$$
$$\lambda_1 f(v) + \lambda_2 g(v) = 0_W$$

Using that $f(u) = g(u)$ and $f(v) = -g(v)$ this yields

$$(\lambda_1 + \lambda_2) f(u) = 0_W$$
$$(\lambda_1 - \lambda_2) f(v) = 0_W$$

As $f(u), f(v) \neq 0_W$, it follows that $\lambda_1 + \lambda_2 = 0, \lambda_1 - \lambda_2 = 0$. This linear system of equations only has the trival solution and therefore f and g are linearly independent.

Possible solution 8b: (i) We show the claim by contradiction. Assume that v_1 and v_2 were linearly dependent. Since both of the vectors are not equal to the zero vector (as $f(v_1) = f(v_2) \neq 0_W$), it follows that there exists a $0 \neq \lambda \in \mathbb{R}$ with $v_2 = \lambda v_1$. Therefore $f(v_2) = f(\lambda v_1) = \lambda f(v_1) = \lambda f(v_2)$ and therefore $\lambda = 1$ since $f(v_2) \neq 0_W$. Thus, $v_2 = v_1$, a contradiction.

(ii) We show the claim by contradiction. Assume that f and g were linearly dependent. As $f(u) = g(u) \neq 0_W$, both f and g are not the zero function. Therefore, there exists a non-zero scalar λ with $f = \lambda g$. It then follows that $f(u) = (\lambda g)(u) = \lambda f(u)$ and therefore $\lambda = 1$ since $f(u) \neq 0_W$. Similarly $f(v) = (\lambda g)(v) = -\lambda f(v)$ and therefore $\lambda = -1$, a contradiction.