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Exam 2021–08–27
Linear algebra II

Writing time: 14:00-19:00. Permitted aids: the course book and your own notes. The exam consists of 8 problems, each problem worth 5 points. A sum of 18 points is required for the grade 3, 25 points for the grade 4, and 32 points for the grade 5. All solutions must contain complete reasoning and not only answers.

Problem 1. Give an example of each of the following, or explain why it does not exist.

(a) A linearly independent subset of $\mathbb{P}_3$ (the vector space of all polynomials of degree at most three) containing 5 polynomials.

(b) A 3×3-matrix that is not invertible and that has the eigenvalues 2, 3 and 5.

(c) An inner product on $\mathbb{R}^2$ with respect to which the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has length 2.

Solution 1.

(a) The vector space $\mathbb{P}_3$ is 4-dimensional. Every linearly independent subset of a 4-dimensional vector space has at most 4 vectors in it. Therefore, there cannot exist a linearly independent subset of $\mathbb{P}_3$ containing 5 polynomials.

(b) A (quadratic) matrix is invertible if and only if all its eigenvalues are non-zero. According to the problem, the eigenvalues 2, 3, and 5 are all non-zero (and a 3×3-matrix can have at most 3 eigenvalues), therefore such a matrix cannot exist.

(c) Every inner product on $\mathbb{R}^2$ is given by $\langle x, y \rangle = x^T A y$ (for all $x, y \in \mathbb{R}^2$) for a symmetric positive definite 2×2-matrix A. Choosing $A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ gives that

$$\left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = \sqrt{\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle} = \sqrt{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \sqrt{4} = 2.$$

This A is obviously symmetric and positive definite since its eigenvalues 4 and 1 are positive.

Problem 2. In the vector space $\mathbb{V}$ of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ we consider the following three subsets:

(a) $\mathbb{U}_1 = \{f \in \mathbb{V} \mid f(2) = f(1)\}$,

(b) $\mathbb{U}_2 = \{f \in \mathbb{V} \mid f(2) = 3\}$,

(c) $\mathbb{U}_3 = \{f \in \mathbb{V} \mid f(2) \geq 0\}$.

For each of these subsets, determine whether it is a subspace. Give proper explanation.

Solution 2.

(a) To prove that $\mathbb{U}_1$ is a subspace we have to show that $\mathbb{U}_1 \neq \emptyset$ and that for all $f, g \in \mathbb{U}_1$ and all $\lambda \in \mathbb{R}$ we have that $f + g \in \mathbb{U}_1$ and $\lambda f \in \mathbb{U}_1$.

- To show that $\mathbb{U}_1 \neq \emptyset$ consider the function f which is constantly zero. It obviously satisfies $f(2) = f(1)$ and therefore is in $\mathbb{U}_1$. Thus $\mathbb{U}_1$ is non-empty.
- Let $f, g \in \mathbb{U}_1$. This means that $f(2) = f(1)$ and $g(2) = g(1)$. The sum of the functions $f + g$ thus satisfies
$$(f+g)(2) = f(2) + g(2) = f(1) + g(1) = (f+g)(1).$$
Therefore $f + g \in \mathbb{U}_1$.
- Let $f \in \mathbb{U}_1$ and $\lambda \in \mathbb{R}$. This means that $f(2) = f(1)$. The scalar multiplication λf thus satisfies
$$(\lambda f)(2) = \lambda(f(2)) = \lambda(f(1)) = (\lambda f)(1).$$
Therefore, $\lambda f \in \mathbb{U}_1$.

(b) Every subspace of a vector space has to contain the zero vector. The zero vector of the space of all continuous functions is the function which is constantly zero. This function does not satisfy $f(2) = 3$, i.e. is not contained in $\mathbb{U}_2$. It follows that $\mathbb{U}_2$ is not a subspace of $\mathbb{V}$.

(c) To show that $\mathbb{U}_3$ is not a subspace, it suffices to give an example for one of the subspace criteria not being satisfied. Consider the function f which is constantly one on all of $\mathbb{R}$. This function is in $\mathbb{U}_3$ as $f(2) = 1 \geq 0$. However, for $\lambda = -1$, the function λf is the function which is constantly -1 on all of $\mathbb{R}$. This function is not in $\mathbb{U}_3$ since $f(2) = -1 \not\geq 0$. Thus, the subspace criterion $\lambda f \in \mathbb{U}_3$ for all $\lambda \in \mathbb{R}$ and all $f \in \mathbb{U}_3$ is not satisfied, and therefore $\mathbb{U}_3$ is not a subspace of $\mathbb{V}$.

Problem 3. Consider the matrix
$$A = \begin{pmatrix} 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 1 \\ 2 & 6 & 2 & 1 \end{pmatrix}$$

(a) Find a basis for the column space of A.

(b) Find a basis for the row space of A.

(c) Does there exist a 3×3-matrix with the same rank as A? If so, give an example.

Solution 3. We first perform Gaussian elimination to the matrix A:
$$\begin{pmatrix} 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 1 \\ 2 & 6 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 2 & 2 \\ 2 & 6 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

(a) A basis of the column space of A is given by the first, second, and fourth column of A as those are the columns containing the pivot elements in the row echelon form of A. So, a basis of the column space of A is given by $\underline{\mathbf{v}} = \left(\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right)$.

(b) As one can see from the row echelon form of A, the matrix A has rank 3, therefore the three rows of A are linearly independent and thus form a basis of A given by $\underline{\mathbf{w}} = \left(\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -3 \end{pmatrix} \right)$.

(c) Yes, every invertible 3×3-matrix has rank 3, for example the identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is one such example.

Problem 4. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

(a) Is A diagonalisable? If it is, determine a basis of $\mathbb{R}^3$ consisting av eigenvectors of A, if it is not, justify your answer.

(b) Is A orthogonally diagonalisable? If it is, determine an orthonormal basis of $\mathbb{R}^3$ consisting av eigenvectors of A, if it is not, justify your answer.

Solution 4.

(a) The eigenvalues of A are equal to the zeroes of its characteristic polynomial.

$$\begin{aligned} \chi_A(\lambda) &= \det \begin{pmatrix} 1-\lambda & 2 & 3 \\ 1 & 2-\lambda & 0 \\ 1 & -1 & -\lambda \end{pmatrix} \\ &= 3\det \begin{pmatrix} 1 & 2-\lambda \\ 1 & -1 \end{pmatrix} + \lambda \det \begin{pmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{pmatrix} \\ &= 3(-1-(2-\lambda)) + (-\lambda)((1-\lambda)(2-\lambda)-2) \\ &= 3(\lambda-3) - \lambda(\lambda^2-3) \\ &= -(\lambda-3)(\lambda^2-3) \end{aligned}$$

The eigenvalues of A are therefore given by 3, $\sqrt{3}$, and $-\sqrt{3}$. As A has three different eigenvalues, it follows that A is diagonalizable. To determine a basis of $\mathbb{R}^3$ consisting of eigenvalues of A we proceed by computing a basis of the corresponding eigenspaces.

- $\lambda = 3$. We do Gaussian elimination:

$$\begin{pmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 1 & -1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

  Therefore the eigenspace to 3 is spanned by $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

- $\lambda = \sqrt{3}$. We do Gaussian elimination:

$$\begin{pmatrix} 1-\sqrt{3} & 2 & 3 \\ 1 & 2-\sqrt{3} & 0 \\ 1 & -1 & -\sqrt{3} \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -\sqrt{3} \\ 0 & 3-\sqrt{3} & \sqrt{3} \\ 0 & 3-\sqrt{3} & \sqrt{3} \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -\sqrt{3} \\ 0 & 3-\sqrt{3} & \sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}.$$

  Therefore the eigenspace to $\sqrt{3}$ is spanned by $\mathbf{v}_2 = \begin{pmatrix} -3+2\sqrt{3} \\ -\sqrt{3} \\ 3-\sqrt{3} \end{pmatrix}$.

- $\lambda = -\sqrt{3}$. We do Gaussian elimination:
$$\begin{pmatrix} 1+\sqrt{3} & 2 & 3 \\ 1 & 2+\sqrt{3} & 0 \\ 1 & -1 & \sqrt{3} \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & \sqrt{3} \\ 0 & 3+\sqrt{3} & -\sqrt{3} \\ 0 & 3+\sqrt{3} & -\sqrt{3} \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & \sqrt{3} \\ 0 & 3+\sqrt{3} & -\sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}.$$
Therefore, the eigenspace to 3 is spanned by $\mathbf{v}_3 = \begin{pmatrix} -3-2\sqrt{3} \\ \sqrt{3} \\ 3+\sqrt{3} \end{pmatrix}$.

A basis of $\mathbb{R}^3$ consisting of eigenvectors of A is therefore given by $\underline{\mathbf{v}} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(b) By the spectral theorem, a matrix is orthogonally diagonalisable if and only if it is symmetric. Since A is not symmetric, it is therefore not orthogonally diagonalisable.

Problem 5.

(a) Determine the base change matrix from the basis $\underline{\mathbf{v}} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$ to the basis $\underline{\mathbf{w}} = \left(\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right)$ of $\mathbb{R}^3$.

(b) Let $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ be the base change matrix from the basis $\underline{\mathbf{u}} = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$ to a basis $\underline{\mathbf{t}}$ of $\mathbb{R}^2$. Determine $\underline{\mathbf{t}}$.

(c) Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map, which with respect to the bases $\underline{\mathbf{w}}$ of $\mathbb{R}^3$ and $\underline{\mathbf{u}}$ of $\mathbb{R}^2$ has the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. What is the matrix of f with respect to the bases $\underline{\mathbf{v}}$ and $\underline{\mathbf{t}}$?

Solution 5.

(a) To compute the base change matrix from $\underline{\mathbf{v}}$ to $\underline{\mathbf{w}}$ we have to express every basis vector in $\underline{\mathbf{v}}$ as a linear combination of the basis vectors in $\underline{\mathbf{w}}$. Doing this for every possibility yields three linear systems of equations which can be summarized in the following matrix (on which we perform Gaussian elimination)
$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -5 & 1 & 1 & -2 & 0 \\ 0 & -10 & 1 & 0 & -4 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -5 & 1 & 1 & -2 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & -\frac{1}{5} & \frac{3}{5} \\ 0 & 1 & 0 & \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right)$$
Therefore, we obtain $T^{\underline{\mathbf{v}}}_{\underline{\mathbf{w}}} = \begin{pmatrix} -\frac{3}{5} & -\frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 2 & 0 & -1 \end{pmatrix}$.

(b) It is convenient to first compute the inverse of A, $T^{\underline{\mathbf{t}}}_{\underline{\mathbf{u}}} = \frac{1}{5}\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$. It follows that

$$\mathbf{t}_1 = \frac{4}{5}\begin{pmatrix}1\\1\end{pmatrix} - \frac{3}{5}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{5}\\\frac{4}{5}\end{pmatrix}$$
$$\mathbf{t}_2 = -\frac{1}{5}\begin{pmatrix}1\\1\end{pmatrix} + \frac{2}{5}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{5}\\-\frac{1}{5}\end{pmatrix}$$

(c) By the base change formula we obtain:

$$(f)^{\underline{\mathbf{v}}}_{\underline{\mathbf{t}}} = T^{\underline{\mathbf{u}}}_{\underline{\mathbf{t}}}(f)^{\underline{\mathbf{w}}}_{\underline{\mathbf{u}}}T^{\underline{\mathbf{v}}}_{\underline{\mathbf{w}}} = \begin{pmatrix}2 & 1\\3 & 4\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}-\frac{3}{5} & -\frac{1}{5} & \frac{3}{5}\\ \frac{1}{5} & \frac{2}{5} & -\frac{1}{5}\\ 2 & 0 & -1\end{pmatrix} = \begin{pmatrix}\frac{17}{5} & \frac{4}{5} & -\frac{7}{5}\\ \frac{28}{5} & \frac{11}{5} & -\frac{13}{5}\end{pmatrix}.$$

Problem 6. Show that the surface given by the equation

$$9x_1^2 + 10x_2^2 + 6x_3^2 + 4x_1x_3 = 20,$$

is an ellipsoid. Determine the points on the ellipsoid which are furthest away from the origin. The coordinates of these points should be given in the original coordinate system.

Solution 6. The surface is an ellipsoid if and only if the eigenvalues of the associated matrix are all positive. We determine the eigenvalues, by looking for the zeroes of the characteristic polynomial of the matrix

$$A = \begin{pmatrix}9 & 0 & 2\\0 & 10 & 0\\2 & 0 & 6\end{pmatrix}$$

We obtain

$$\begin{aligned}\chi_A(\lambda) &= \det\begin{pmatrix}9-\lambda & 0 & 2\\0 & 10-\lambda & 0\\2 & 0 & 6-\lambda\end{pmatrix}\\ &= (10-\lambda)\det\begin{pmatrix}9-\lambda & 2\\2 & 6-\lambda\end{pmatrix}\\ &= (10-\lambda)((9-\lambda)(6-\lambda)-4)\\ &= (10-\lambda)(\lambda^2-15\lambda+50)\end{aligned}$$

Therefore, the eigenvalues are 10 and $\frac{15}{2} \pm \sqrt{\frac{225}{4} - 50} = \begin{cases}10,\\5.\end{cases}$ The points furthest away from the origin lie on the principal axis corresponding to the smallest eigenvalue. We compute the corresponding eigenspace using Gaussian elimination:

$$\begin{pmatrix}4 & 0 & 2\\0 & 5 & 0\\2 & 0 & 1\end{pmatrix} \sim \begin{pmatrix}4 & 0 & 2\\0 & 5 & 0\\0 & 0 & 0\end{pmatrix}$$

Therefore, the corresponding eigenspace is spanned by $\begin{pmatrix}1\\0\\-2\end{pmatrix}$. The length of the principal axis is $\sqrt{\frac{20}{5}} = 2$ and therefore the points furthest away from the origin are those of length 2 on the line through $\begin{pmatrix}1\\0\\-2\end{pmatrix}$. These are given by $\pm\frac{2}{\sqrt{5}}(1, 0, -2)$.

Problem 7. Determine the solution to the following system of linear differential equations

$$\begin{cases} y_1' &= -2y_1, \\ y_2' &= 5y_1 + 3y_2, \\ y_3' &= 11y_1 + 2y_2 + y_3. \end{cases}$$

which satisfies $y_1(0) = 1$, $y_2(0) = 1$, $y_3(0) = 1$.

Solution 7. We write the equation in matrix form $y' = Ay$ for $A = \begin{pmatrix} -2 & 0 & 0 \\ 5 & 3 & 0 \\ 11 & 2 & 1 \end{pmatrix}$. As A is lower triangular, its eigenvalues are given by its diagonal entries, -2, 3, and 1. We proceed by determining bases of the corresponding eigenspaces.

- For $\lambda = -2$: $\begin{pmatrix} 0 & 0 & 0 \\ 5 & 5 & 0 \\ 11 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -9 & 3 \\ 0 & 0 & 0 \end{pmatrix}$. Thus, the vector $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ spans the eigenspace to the eigenvalue -2.
- For $\lambda = 3$: $\begin{pmatrix} -5 & 0 & 0 \\ 5 & 0 & 0 \\ 11 & 2 & -2 \end{pmatrix}$. Thus, the vector $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ spans the eigenspace to the eigenvalue 3.
- For $\lambda = 1$: One can see from the matrix, that the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ spans the corresponding eigenspace.

Substituting $y = Sz$ for $S = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$ we obtain the system $z' = Dz$ for the diagonal matrix $D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This system has the solution $z = \begin{pmatrix} C_1 e^{-2t} \\ C_2 e^{3t} \\ C_3 e^t \end{pmatrix}$ and therefore $y = Sz = \begin{pmatrix} -C_1 e^{-2t} \\ C_1 e^{-2t} + C_2 e^{3t} \\ 3C_1 e^{-2t} + C_2 e^{3t} + C_3 e^t \end{pmatrix}$. Taking the initial condition into account we obtain the additional condition that

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -C_1 \\ C_1 + C_2 \\ 3C_1 + C_2 + C_3 \end{pmatrix}.$$

Solving this yields $C_1 = -1, C_2 = 2, C_3 = 3$, so in summary the unique solution is given by

$$y = \begin{pmatrix} e^{-2t} \\ -e^{-2t} + 2e^{3t} \\ -3e^{-2t} + 2e^{3t} + 2e^t \end{pmatrix}$$

Problem 8. Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0 \right\}$. Define an addition on V via

- $\begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} x' \\ 0 \end{pmatrix} = \begin{pmatrix} x + x' \\ 0 \end{pmatrix}$,

- $\begin{pmatrix}0\\y\end{pmatrix} + \begin{pmatrix}0\\y'\end{pmatrix} = \begin{pmatrix}0\\y+y'\end{pmatrix}$,
- $\begin{pmatrix}x\\0\end{pmatrix} + \begin{pmatrix}0\\y\end{pmatrix} = \begin{pmatrix}0\\y\end{pmatrix} + \begin{pmatrix}x\\0\end{pmatrix} = \begin{pmatrix}x+y\\0\end{pmatrix}$ for $x \neq 0$ and $y \neq 0$.

Which of the vector space axioms are satisfied for V with this addition and the normal scalar multiplication?

Solution 8. As addition and scalar multiplication are obviously well-defined, there are eight axioms to check in total.

- Associativity of addition $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ does not hold for all vectors. As a counterexample consider

$$\begin{pmatrix}1\\0\end{pmatrix} + \left(\begin{pmatrix}-1\\0\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix}\right) = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$
$$\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-1\\0\end{pmatrix}\right) + \begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$$

- Commutativity of addition $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ does hold for all vectors. There are three cases to consider:
  - For $\mathbf{v} = \begin{pmatrix}x\\0\end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix}x'\\0\end{pmatrix}$ we have
  $$\mathbf{v} + \mathbf{w} = \begin{pmatrix}x+x'\\0\end{pmatrix} = \begin{pmatrix}x'+x\\0\end{pmatrix} = \mathbf{w} + \mathbf{v}.$$
  - For $\mathbf{v} = \begin{pmatrix}0\\y\end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix}0\\y'\end{pmatrix}$ we have
  $$\mathbf{v} + \mathbf{w} = \begin{pmatrix}0\\y+y'\end{pmatrix} = \begin{pmatrix}0\\y'+y\end{pmatrix} = \mathbf{w} + \mathbf{v}.$$
  - For $\mathbf{v} = \begin{pmatrix}x\\0\end{pmatrix}$ and $w = \begin{pmatrix}0\\y\end{pmatrix}$ we have
  $$\mathbf{v} + \mathbf{w} = \begin{pmatrix}x+y\\0\end{pmatrix} = \mathbf{w} + \mathbf{v}.$$

- The zero vector is given by $\begin{pmatrix}0\\0\end{pmatrix}$ as for the two cases we have $\begin{pmatrix}x\\0\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}x\\0\end{pmatrix}$ and $\begin{pmatrix}0\\y\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\y\end{pmatrix}$.

- For the existence of additive inverses, we show that $-\begin{pmatrix}x\\0\end{pmatrix} = \begin{pmatrix}-x\\0\end{pmatrix}$ and $-\begin{pmatrix}0\\y\end{pmatrix} = \begin{pmatrix}0\\-y\end{pmatrix}$ as $\begin{pmatrix}x\\0\end{pmatrix} + \begin{pmatrix}-x\\0\end{pmatrix} = \begin{pmatrix}x+(-x)\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\y\end{pmatrix} + \begin{pmatrix}0\\-y\end{pmatrix} = \begin{pmatrix}0\\y+(-y)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

- Associativity of scalar multiplication $(\lambda\mu)\mathbf{v} = \lambda(\mu\mathbf{v})$ does not involve the changed addition but only the normal scalar multiplication and therefore does hold.

- $1 \cdot \mathbf{v}$ also involves only scalar multiplication and therefore does hold.
- The first distributivity $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$ does hold. There are three cases to consider:
  - For $\mathbf{v} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} x' \\ 0 \end{pmatrix}$ we obtain
  $$\lambda(\mathbf{v}+\mathbf{w}) = \lambda \begin{pmatrix} x + x' \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda(x + x') \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda x + \lambda x' \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda x \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda x' \\ 0 \end{pmatrix} = \lambda\mathbf{v}+\lambda\mathbf{w}.$$
  - For $\mathbf{w} = \begin{pmatrix} 0 \\ y \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ y' \end{pmatrix}$ we obtain
  $$\lambda(\mathbf{v}+\mathbf{w}) = \lambda \begin{pmatrix} 0 \\ y + y' \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda(y + y') \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda y + \lambda y' \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda y \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda y' \end{pmatrix} = \lambda\mathbf{v}+\lambda\mathbf{w}.$$
  - For $\mathbf{v} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ y \end{pmatrix}$ for $x, y \neq 0$ we obtain
  $$\lambda(\mathbf{v} + \mathbf{w}) = \lambda \begin{pmatrix} x + y \\ 0 \end{pmatrix} = \big(\lambda(x + y)\big) = \big(\lambda x + \lambda y\big) = \begin{pmatrix} \lambda x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda y \end{pmatrix} = \lambda\mathbf{v} + \lambda\mathbf{w}.$$
- The second distributivity $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$ holds. There are two cases to consider:
  - For $\mathbf{v} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ we obtain
  $$(\lambda + \mu)\mathbf{v} = \begin{pmatrix} (\lambda + \mu)x \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda x + \mu x \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda x \\ 0 \end{pmatrix} + \begin{pmatrix} \mu x \\ 0 \end{pmatrix} = \lambda\mathbf{v} + \mu\mathbf{v}.$$
  - For $\mathbf{v} = \begin{pmatrix} 0 \\ y \end{pmatrix}$ we obtain
  $$(\lambda + \mu)\mathbf{w} = \begin{pmatrix} 0 \\ (\lambda + \mu)y \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda y + \mu y \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda y \end{pmatrix} + \begin{pmatrix} 0 \\ \mu y \end{pmatrix} = \lambda\mathbf{v} + \mu\mathbf{v}.$$

We have checked that all vector space axioms except for associativity of addition hold in this example.