solution

Thursday, March 3, 2022 2:18 PM

1. Each of the following questions must be answered with YES or NO. If the answer is YES, an example must be given — not an explanation. If the answer is NO, an explanation is needed.
   (a) Is there a surjective linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ which is not injective?

   Answer: NO. If F is a surjective linear map, then $\dim \mathrm{Im} F = 2$. By the dimension theorem $\dim \mathrm{Ker}\, F = 2 - \dim \mathrm{Im}F = 0$, so $\mathrm{Ker}\, F = 0$, which means F is injective.

   (b) Is there a symmetric 2×2 matrix that is not positive definite?

   Answer: YES, e.g., the zero matrix.

   (c) Is there a 2×2 matrix that is not diagonalisable?

   Answer: YES, e.g., $\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$ has no real eigenvalue so is not diagonalisable.

2. Determine whether the subset $\mathbb{U}$ is a subspace of the vector space $\mathbb{V}$. Justify your answer.
   (a) $\mathbb{V} = \mathbb{P}_2$ and $\mathbb{U} = \{p \in \mathbb{V} \mid p'(x) = x\}$.

   Answer: The zero element does not belong to $\mathbb{U}$, so $\mathbb{U}$ is not a subspace.

   (b) $\mathbb{V} = M_{2\times 2}(\mathbb{R})$ and $\mathbb{U} = \{A \in M_{2\times 2}(\mathbb{R}) \mid A = A^t\}$.

   Answer: It is a subspace because:
   1. $0^t = 0$, so $0 \in \mathbb{U}$. In particular, $\mathbb{U}$ is nonempty.
   2. If $A, B \in \mathbb{U}$, then $(A + B)^t = A^t + B^t = A + B$, and thus $A + B \in \mathbb{U}$.
   3. If $A \in \mathbb{U}$ and $\lambda \in R$ then we have $(\lambda A)t = \lambda A^t = \lambda A$, so $\lambda A \in \mathbb{U}$.

3. Let $G : \mathbb{P}_2 \to \mathbb{P}_1$ be the linear map where

$$G(x^2) = x, \quad G(x) = x + 1, \quad G(1) = x$$

and let $H : \mathbb{P}_1 \to \mathbb{R}^2$ be the linear map given by $q \mapsto (q'(0), q(0))$. Consider the bases $\underline{\mathbf{v}} = (1, x, x^2)$ of $\mathbb{P}_2$, $\underline{\mathbf{u}} = (1, x)$ of $\mathbb{P}_1$ and $\underline{\mathbf{e}} = (\mathrm{e}_1, \mathrm{e}_2)$ of $\mathbb{R}^2$.
   (a) Find $G^{\underline{\mathbf{v}}}_{\underline{\mathbf{u}}}$ and $H^{\underline{\mathbf{u}}}_{\underline{\mathbf{e}}}$.
   (b) Is the composition $H \circ G$ a linear map? If yes, find the matrix $(H \circ G)^{\underline{\mathbf{v}}}_{\underline{\mathbf{e}}}$; if not, explain why.
   (c) Is there a linear map $F : \mathbb{P}_2 \to \mathbb{P}_1$ with $F \neq G$ such that $H \circ F = H \circ G$? If yes, give an example of such F; if not, explain why.

(a) $G^{\underline{v}}_{\underline{u}} = \left(G(1)_{\underline{u}} \quad G(x)_{\underline{u}} \quad G(x^2)_{\underline{u}} \right) = \left((x)_{\underline{u}} \quad (x+1)_{\underline{u}} \quad (x)_{\underline{u}} \right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$H^{\underline{u}}_{\underline{e}} = \left(H(1)_{\underline{e}} \quad H(x)_{\underline{e}} \right) = \begin{pmatrix} 1'|_{x=0} & x'|_{x=0} \\ 1|_{x=0} & x|_{x=0} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(b) Yes, $(H \circ G)^{\underline{v}}_{\underline{e}} = H^{\underline{u}}_{\underline{e}} G^{\underline{v}}_{\underline{u}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(c) No, since H is invertible: if $H \circ F = H \circ G$ then

$H^{-1} \circ H \circ F = H^{-1} \circ H \circ G = Id \circ G = G$.

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$$F = \overset{\shortparallel}{\mathrm{Id} \circ F}$$

4. Let

$$A = \begin{pmatrix} 4 & 4 & 9 & 0 & 35 \\ -1 & -1 & -2 & 0 & -8 \\ -3 & -3 & -7 & 0 & -27 \\ 1 & 1 & 2 & 0 & 8 \end{pmatrix}$$

(a) Find a basis of the column space of A.

(b) What is the dimension of the row space of A?

(c) What is the dimension of the nullspace (kernel) of A?

$$\begin{matrix} 4 & 4 & 9 & 0 & 35 \\ -1 & -1 & -2 & 0 & -8 \\ -3 & -3 & -7 & 0 & -27 \\ 1 & 1 & 2 & 0 & 8 \end{matrix} \begin{matrix} +4 \\ \\ -3 \\ +1 \end{matrix} \rightsquigarrow \begin{matrix} 0 & 0 & 1 & 0 & 3 \\ -1 & -1 & -2 & 0 & -8 \\ 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \begin{matrix} \\ \times -1 \\ +1 \\ \\ \end{matrix} \rightsquigarrow \begin{matrix} 0 & 0 & 1 & 0 & 3 \\ 1 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$$\rightsquigarrow \begin{matrix} 1 & 1 & 2 & 0 & 8 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

1st and 3rd columns form a basis to column space.

(a) $\left(\begin{pmatrix} 4 \\ -1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 9 \\ -2 \\ -7 \\ 2 \end{pmatrix} \right)$

(b) $\dim(\text{row space}) \underset{\text{rank theorem}}{=} \dim(\text{column space}) \underset{\text{by (a)}}{=} 2$

Alternatively, the above computation gives a basis to the row space which has two elements.

(c) $\dim \ker A = 5 - \dim(\text{column space}) = 5 - 2 = 3$

5. Equip $\mathbb{R}^3$ with the standard scalar product and consider the subspace $\mathbb{U} = [(1,1,0),(1,2,2)]$ in $\mathbb{R}^3$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the projection to $\mathbb{U}$.

(a) Find an orthonormal (ON) basis in $\mathbb{U}$.

(b) Find an ON basis in $\mathbb{U}^\perp$.

(c) Find the matrix of F in an ON basis of your choice.

(d) Find the matrix of F in the standard basis.

(a) $\left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}\right)$ is a basis for $\mathbb{U}$, so we apply Gram-Schmidt process to get an ON basis:

$$w_1 = \frac{v_1}{|v_1|} = \frac{1}{\sqrt{1^2+1^2+0^2}} v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\tilde{w}_2 = (v_2)_{\perp[v_1]} = v_2 - (v_2|w_1)w_1 = \begin{pmatrix}1\\1\\2\end{pmatrix} - \frac{1}{\sqrt{2}}(1\cdot1+1\cdot2+2\cdot0)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1\\1\\4\end{pmatrix}$$

$$w_2 = \frac{\tilde{w}_2}{|\tilde{w}_2|} = \frac{1}{\sqrt{18}}\begin{pmatrix}-1\\1\\4\end{pmatrix}$$

Answer: $\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \frac{1}{\sqrt{18}}\begin{pmatrix}-1\\1\\4\end{pmatrix}\right)$

(b) $v_3 := \begin{pmatrix}2\\-2\\1\end{pmatrix}$ then $(v_3|v_1) = 0$ & $(v_3|v_2) = 0$ so $v_3 \in \mathbb{U}^\perp$

$\dim \mathbb{U}^\perp = 1$, so $w_3 = \frac{v_3}{|v_3|} = \frac{1}{\sqrt{9}}\begin{pmatrix}2\\-2\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix}$ forms an ONB in $\mathbb{U}^\perp$

(c) We take the ONB $(w_1, w_2, w_3) = \underline{w}$

since $F(w_1) = w_1$, $F(w_2) = w_2$, $F(w_3) = 0$

$$[F]^{\underline{w}}_{\underline{w}} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

(d) Basechange: $(F)^{\underline{e}}_{\underline{e}} = T^{\underline{w}}_{\underline{e}} (F)^{\underline{w}}_{\underline{w}} T^{\underline{e}}_{\underline{w}}$

$$T^{\underline{w}}_{\underline{e}} = \frac{1}{3\sqrt{2}}\begin{pmatrix}3&-1&2\sqrt{2}\\3&1&-2\sqrt{2}\\0&4&\sqrt{2}\end{pmatrix}$$

$$T^{\underline{e}}_{\underline{w}} = {T^{\underline{w}}_{\underline{e}}}^{-1} \underset{\text{ON}}{=} {T^{\underline{w}}_{\underline{e}}}^{t} = \frac{1}{3\sqrt{2}}\begin{pmatrix}3&3&0\\-1&1&4\\2\sqrt{2}&-2\sqrt{2}&\sqrt{2}\end{pmatrix}$$

$$= \frac{1}{18}\begin{pmatrix}3&-1&2\sqrt{2}\\3&1&-2\sqrt{2}\\0&4&\sqrt{2}\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}3&3&0\\-1&1&4\\2\sqrt{2}&-2\sqrt{2}&\sqrt{2}\end{pmatrix}$$

$$= \frac{1}{18}\begin{pmatrix}3&-1&0\\3&1&0\\0&4&0\end{pmatrix}\begin{pmatrix}3&3&0\\-1&1&4\\2\sqrt{2}&-2\sqrt{2}&\sqrt{2}\end{pmatrix} = \frac{1}{18}\begin{pmatrix}10&8&-4\\8&10&4\\-4&4&16\end{pmatrix}$$

6. Let the vector space $\mathbb{V} = M_{2\times2}(\mathbb{R})$ be equipped with a scalar product $(-|-)$ whose matrix in the basis $\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$ is

$$\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&4\end{pmatrix}.$$

(a) Find the length of $\begin{pmatrix}0&0\\1&1\end{pmatrix} \in \mathbb{V}$.

Answer: The coordinate vector of $C := \begin{pmatrix}0&0\\1&1\end{pmatrix} \in \mathbb{V}$ in the given basis is $(0,0,1,1)$. So $|C|^2 =$

6. Let the vector space $\mathbb{V} = M_{2\times 2}(\mathbb{R})$ be equipped with a scalar product $(-|-)$ whose matrix in the basis $\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$ is

$$\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&4\end{pmatrix}.$$

(a) Find the length of $\begin{pmatrix}0&0\\1&1\end{pmatrix} \in \mathbb{V}$.

Answer: The coordinate vector of $C := \begin{pmatrix}0&0\\1&1\end{pmatrix} \in \mathbb{V}$ in the given basis is $(0,0,1,1)$. So $|C|^2 = (C|C) = 1\cdot 0 + 1\cdot 0 + 3\cdot 1^2 + 4\cdot 1^2 = 7$. The length of C is thus $\sqrt{7}$.

(b) For which $A \in \mathbb{V}$ is the map $F : \mathbb{V} \to \mathbb{R}$ given by $F(B) = (B|A)$ a linear map?

Answer: By the bilinearity of scalar product, the map F is a linear map for all $A \in V$.

(c) For which $A \in \mathbb{V}$ is the map $F : \mathbb{V} \to \mathbb{R}$ given by $F(B) = (B|A)$ an isometry? (Here, $\mathbb{R}$ is equipped with the standard scalar product.)

Answer: Since an isometry is injective, there is no isometry from $\mathbb{V}$, which is of dimension 4, to $\mathbb{R}$, of dimension one (e.g., by dimension theorem). So for no $A \in \mathbb{V}$ the map F is an isometry.

7. Consider the map $Q : \mathbb{R}^3 \to \mathbb{R}$ given by

$$Q(\mathbf{x}) = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3,$$

where $\mathbf{x} = (x_1, x_2, x_3)$.

(a) Which type of surface in $\mathbb{R}^3$ does $Q(\mathbf{x}) = 22$ give?

(b) Let d be the distance from a point on the surface to the origin. What is the largest value d can take? What is the smallest value d can take?

(a) Write $Q(x) = (x_1\ x_2\ x_3)\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x^t A x$

Note $A = \begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}$ is symmetric

$$\chi_A(\lambda) = \begin{vmatrix}2-\lambda&1&1\\1&2-\lambda&1\\1&1&2-\lambda\end{vmatrix} = \begin{vmatrix}4-\lambda&4-\lambda&4-\lambda\\1&2-\lambda&1\\1&1&2-\lambda\end{vmatrix} = (4-\lambda)\begin{vmatrix}1&1&1\\1&2-\lambda&1\\1&1&2-\lambda\end{vmatrix}$$

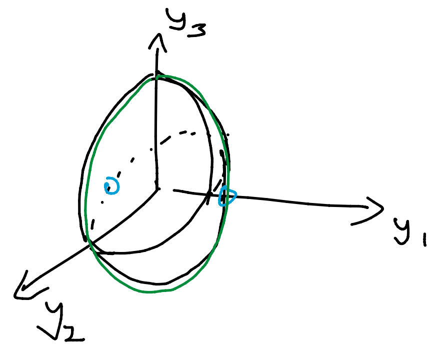
$$= (4-\lambda)\begin{vmatrix}1&1&1\\0&1-\lambda&0\\0&0&1-\lambda\end{vmatrix} = (4-\lambda)(1-\lambda)^2$$

spectral thm

$\Rightarrow$ $\exists$ ON basis $\underline{v}$ s.t. $[A]_{\underline{v}}^{\underline{v}} = \begin{pmatrix}4&0&0\\0&1&0\\0&0&1\end{pmatrix}$

Thus, for some (ON) coordinates $y_1\ y_2\ y_3$, the surface is given by $4y_1^2 + y_2^2 + y_3^2 = 22$ and thus is an ellipsoid.

$$4y_1^2 + y_2^2 + y_3^2 = 22$$ and thus is an ellipsoid.



(b)

Smallest $d = \sqrt{\frac{22}{4}}$ Largest $d = \sqrt{22}$

8. Consider the following system of differential equations.

$$\begin{cases} y_1' = y_1 + 4y_2, \\ y_2' = 2y_1 + 3y_2, \end{cases}$$

(a) Find all solutions to the system.

(b) Find the solution that satisfies $y_1(0) = -1$, $y_2(0) = 2$.

(a) Write $y' = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} y$

$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ has $\chi_A(\lambda) = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix}$

$$= (1-\lambda)(3-\lambda) - 4 \cdot 2$$
$$= \lambda^2 - 4\lambda + 3 - 8$$
$$= (\lambda - 5)(\lambda + 1)$$

So A is diagonalizable and we can solve the system!

Let's find an eigenbasis $\underline{v} = (v_1, v_2)$

$v \in V$ is eigenvector for $\lambda_1 = 5 \iff \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} v = 0$

Take $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

" $\lambda_2 = -1 \iff \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} v = 0$

Take $v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\binom{\ }{-1}$

Then $D = T^{-1} A T = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$.

where $T = T^{v}_{\underline{e}} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$.

So for $z = T^{-1} y$, we have

$$T z' = y' = A T z \iff z' = T^{-1} A T z = D z$$

$$\iff \begin{cases} z_1' = 5 z_1 \\ z_2' = -z_2 \end{cases}$$

The latter is solved as $\begin{cases} z_1 = c_1 e^{5t} \\ z_2 = c_2 e^{-t} \end{cases}$ for $c_1, c_2 \in \mathbb{R}$.

The solutions are $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = T z = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{5t} + 2 c_2 e^{-t} \\ c_1 e^{5t} - c_2 e^{-t} \end{pmatrix}$

for $c_1, c_2 \in \mathbb{R}$

(b) Condition says

$$y_1(0) = c_1 + 2c_2 = -1$$

$$y_2(0) = c_1 - c_2 = 2,$$

so $c_1 = 1$, $c_2 = -1$.

Thus the solution is $y_1(t) = e^{5t} - 2e^{-t}$

$y_2(t) = e^{5t} + e^{-t}$