LINEAR ALGEBRA III EXAM

Course: 1MA026 **Time:** 2012-03-07 8:00-13:00

Only writing tools are allowed. Solutions may be written in Swedish or English. Motivate your answers carefully. Each problem is worth 5 points. For grade 3/4/5 you will need 18/25/32 (including bonus points from the assignments) of the total score. Good luck!

1. Compute $\cos(T)$ and $\sin(T)$ and e^{iT} where T is the nilpotent matrix

$$T = \begin{bmatrix} 2 & 8 & -2 \\ -1 & -4 & 1 \\ -1 & -6 & 2 \end{bmatrix}.$$

2. We define an inner product on the vector space $\mathcal{C}[-1,1]$ of continuous functions $[-1,1] \to \mathbb{R}$ by $\langle f(x), g(x) \rangle = \frac{1}{2}\int_{-1}^{1} f(x)g(x)dx$.

   a) Find an orthonormal basis of the subspace $\mathcal{P}_2$ of polynomials with degree ≤ 2.
   b) Find the function in $\mathcal{P}_2$ closest to x^3+1 with respect to our chosen inner product.

3. Give the definition of the characteristic polynomial and the minimal polynomial of a square matrix.

4. Find an invertible matrix S and a matrix J in Jordan form such that $S^{-1}AS = J$ where

$$A = \begin{bmatrix} -6 & -8 & -8 \\ 2 & 2 & 3 \\ 4 & 4 & 6 \end{bmatrix}.$$

5. Let $\mathcal{P}_6$ be the real vector space of polynomials with real coefficents and degree less than or equal to 6. Let ∂ be the differentiation operator on $\mathcal{P}_6$: $\partial(p(t)) = p'(t)$.

   a) Show that $\mathcal{S} = \{p \in \mathcal{P}_6 \mid p(t) + p(-t) = 0\}$ and $\mathcal{T} = \{p \in \mathcal{P}_6 \mid p(t) - p(-t) = 0\}$ are both subspaces of $\mathcal{P}_6$.
   b) Show that $\mathcal{S}$ and $\mathcal{T}$ are both invariant under ∂^2.
   c) Show that $\mathcal{P}_6 = \mathcal{S} \oplus \mathcal{T}$.

6. Let V be a complex inner product space with orthonormal basis $\{e_1, e_2, e_3\}$. Let φ be an operator on V such that $\varphi(e_1) = e_1 + e_3$, $\varphi(e_2) = e_1 + e_2$, $\varphi(e_3) = e_2 + e_3$. Does there exist an orthonormal basis of V consisting of eigenvectors of φ? Motivate your answer.

7. Prove that all the eigenvalues of a self-adjoint operator on a complex inner product space are real numbers.

8. a) Define addition and scalar multiplication on $\mathbb{R}^+$, the set of positive real numbers, so that it becomes a real vector space with additive identity 1.
   b) Is it possible to define addition and scalar multiplication on $\mathbb{Q}$ so that it becomes a $\mathbb{Q}$-vector space of dimension 2? Motivate your answer.