SOLUTIONS

LINEAR ALGEBRA III EXAM

Course: 1MA026 **Time:** 2012-03-07 8:00-13:00

1. Compute $\cos(T)$ and $\sin(T)$ and e^{iT} where T is the nilpotent matrix

$$T = \begin{bmatrix} 2 & 8 & -2 \\ -1 & -4 & 1 \\ -1 & -6 & 2 \end{bmatrix}.$$

Suggested solution:
We know $T^3 = 0$ since the nilpotency degree can't exceed the dimension. We find that

$$T^2 = \begin{bmatrix} -2 & -4 & 0 \\ 1 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix}.$$

We have $\sin(x) = x - x^3/6 + \ldots$ and $\cos(x) = 1 - x^2/2 + \ldots$ so

$$\sin(T) = T \qquad cos(T) = I - T^2/2$$

since all powers higher than two are zero. Since $e^{ix} = \cos(x) + i\sin(x)$ we get

$$e^{iT} = \cos(T) + i\sin(T) = I + iT - T^2/2$$

(alternativly one could compute e^{iT} directly). Conclusion:

$$\cos(T) = \frac{1}{2}\begin{bmatrix} 4 & 4 & 0 \\ -1 & 0 & 0 \\ -2 & -4 & 2 \end{bmatrix}, \qquad \sin(T) = \begin{bmatrix} 2 & 8 & -2 \\ -1 & -4 & 1 \\ -1 & -6 & 2 \end{bmatrix},$$

$$e^{iT} = \frac{1}{2}\begin{bmatrix} 4+4i & 4+16i & -4i \\ -1-2i & -8i & 2i \\ -2-2i & -4-12i & 2+4i \end{bmatrix}.$$

2. We define an inner product on the vector space $\mathcal{C}[-1,1]$ of continuous functions $[-1,1] \to \mathbb{R}$ by $\langle f(x), g(x)\rangle = \frac{1}{2}\int_{-1}^{1} f(x)g(x)dx$.

   a) Find an orthonormal basis of the subspace $\mathcal{P}_2$ of polynomials with degree ≤ 2.
   b) Find the function in $\mathcal{P}_2$ closest to x^3+1 with respect to our chosen inner product.

Suggested solution:

a) We have the standard basis $\{1, x, x^2\}$ of $\mathcal{P}_2$ and we can use Gram-Schmidt to make it orthonormal. $\langle 1, 1\rangle = 1$ so we take

$$e_1 = 1.$$

$\langle 1, x\rangle = 0$ so we just need to normalize x. $\langle x, x\rangle = \frac{1}{3}$ so we take

$$e_2 = \frac{x}{||x||} = \sqrt{3}x.$$

Finally, $\langle x^2, x\rangle = 0$ and $\langle x^2, 1\rangle = \frac{1}{3}$ so $e_3' = x^2 - \langle x^2, 1\rangle 1 - \langle x^2, \sqrt{3}x\rangle\sqrt{3}x = x^2 - \frac{1}{3}$ is orthogonal to both e_1 and e_2. Finally, we normalize: $\langle e_3', e_3'\rangle = \frac{4}{45}$, so we take

$$e_3 = \frac{e_3'}{||e_3'||} = \frac{\sqrt{5}}{2}(3x^2 - 1)$$

Conclusion:

$$\{1,\ \sqrt{3}x,\ \frac{\sqrt{5}}{2}(3x^2-1)\}$$

is an orthonormal basis of $\mathcal{P}_2$.

b) We project $x^3 + 1$ onto our orthonormal basis.

$$\begin{aligned} Proj_{\mathcal{P}_2}(x^3+1) &= \langle x^3+1, 1\rangle\ 1 + \langle x^3+1, \sqrt{3}x\rangle\sqrt{3}x + \langle x^3+1, \frac{\sqrt{5}}{2}(3x^2-1)\rangle\frac{\sqrt{5}}{2}(3x^2-1) \\ &= 1\cdot 1 + \frac{\sqrt{3}}{5}\cdot\sqrt{3}x + 0\cdot\frac{\sqrt{5}}{2}(3x^2-1) = \frac{3}{5}x + 1. \end{aligned}$$

Thus $\frac{3}{5}x + 1$ is the function in $\mathcal{P}_2$ closest to $x^3 + 1$ with respect to our chosen inner product.

3. Give the definition of the characteristic polynomial and the minimal polynomial of a square matrix.

Suggested solution:
Let A be a square matrix.
The characteristic polynomial of A is defined as $p_A(t) = det(tI - A)$.

The minimal polynomial $\mu(A)$ is the unique monic polynomial of minimal degree which annihilates A.

4. Find an invertible matrix S and a matrix J in Jordan form such that $S^{-1}AS = J$ where
$$A = \begin{bmatrix} -6 & -8 & -8 \\ 2 & 2 & 3 \\ 4 & 4 & 6 \end{bmatrix}.$$

Suggested solution:
The characteristic polynomial of A is
$$p_A(t) = \det(tI - A) = t^2(t-2).$$

We first consider the eigenvalue 2. Since the multiplicity of 2 in the characteristic polynomial is 1, we will have a single Jordan block of size 1 corresponding to the eigenvalue 2. We find an eigenvector $u_1 = (-3, 1, 2)$ by solving $(A - 2I)u = 0$.

Now consider the eigenvalue 0. In this case, $(A - 0I) = A$ so we start by finding the kernel and image of A. We obtain
$$ker\ A = span\{(-4,1,2)\} \qquad Im\ A = span\{(1,0,0),\ (0,1,2)\}.$$

Since the kernel is one-dimensional the Jordan form will contain a single block of size 2 corresponding to the eigenvalue 0. Thus the first vector of the Jordan chain is given by any nonzero vector of $ker\ A \cap Im\ A = span\{(-4,1,2)\}$. Thus we take $v_1 = (-4,1,2)$. Solving $(A - 0I)v_2 = v_1$ we obtain $v_2 = (2, 0, -1)$. We arrange u_1, v_1, v_2 as columns of a matrix S, and we put the two Jordan blocks correspondingly in a matrix J. Thus with
$$S = [u_1\ \ v_1\ \ v_2] = \begin{bmatrix} -3 & -4 & 2 \\ 1 & 1 & 0 \\ 2 & 2 & -1 \end{bmatrix} \qquad J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$
we do indeed have $S^{-1}AS = J$.

Note that although J is unique up to permutation of the two blocks, there are many other valid choices for the matrix S.

5. Let $\mathcal{P}_6$ be the real vector space of polynomials with real coefficents and degree less than or equal to 6. Let ∂ be the differentiation operator on $\mathcal{P}_6$: $\partial(p(t)) = p'(t)$.

a) Show that $\mathcal{S} = \{p \in \mathcal{P}_6 \mid p(t) + p(-t) = 0\}$ and $\mathcal{T} = \{p \in \mathcal{P}_6 \mid p(t) - p(-t) = 0\}$ are both subspaces of $\mathcal{P}_6$.
b) Show that $\mathcal{S}$ and $\mathcal{T}$ are both invariant under ∂^2.
c) Show that $\mathcal{P}_6 = \mathcal{S} \oplus \mathcal{T}$.

Suggested solution:
a) Let $p, q \in \mathcal{S}$. Then

$$(p+q)(t) + (p+q)(-t) = (p(t) + p(-t)) + (q(t) + q(-t)) = 0 + 0 = 0,$$

so $p + q \in \mathcal{S}$ and

$$(\lambda p)(t) + (\lambda p)(-t) = \lambda(p(t) + p(-t)) = \lambda 0 = 0,$$

so $\lambda p \in \mathcal{S}$ and $\mathcal{S}$ is a subspace. Analogously one shows that $\mathcal{T}$ is a subspace.

We recognize $\mathcal{S}$ and $\mathcal{T}$ as the subspaces of odd functions and even functions in $\mathcal{P}_6$.

b) Let $p \in \mathcal{S}$. Then $0 = p(t) + p(-t)$, so by differentiating twice we get

$$0 = \partial^2(p(t) + p(-t)) = p''(t) + (-1)^2 p''(-t) = (\partial^2 p)(t) + (\partial^2 p)(-t),$$

which means $(\partial^2 p) \in \mathcal{S}$, and $\mathcal{S}$ is ∂^2-invariant. The same argument works for $\mathcal{T}$.

c) Any function $f : \mathbb{R} \to \mathbb{R}$ can be written $f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$. One easily checks that the first term is an even function and the second term is an odd function. In particular, this shows that $\mathcal{S} + \mathcal{T} = \mathcal{P}_6$.

Let $p \in \mathcal{S} \cap \mathcal{T}$. Then $p(t) + p(-t) = 0 = p(t) - p(-t)$ for all $t \in \mathbb{R}$. Subtracting $p(t)$ from each side gives $p(-t) = -p(-t)$ for all $t \in \mathbb{R}$, which implies $p(t) = 0$ for all $t \in \mathbb{R}$, so p is the zero polynomial.

Thus $\mathcal{S} + \mathcal{T} = \mathcal{P}_6$ and $\mathcal{S} \cap \mathcal{T} = \{0\}$ which means $\mathcal{P}_6 = \mathcal{S} \oplus \mathcal{T}$.

Remark: The above proofs for statements b) and c) never uses the fact that we are working with polynomials. Indeed the same statements are true in the subspace of C^∞-smooth functions. In the case of $\mathcal{P}_6$ however, one could also prove the statements directly by noting that $\{x, x^3, x^5\}$ is a basis for $\mathcal{S}$ and that $\{1, x^2, x^4, x^6\}$ is a basis for $\mathcal{T}$.

6. Let V be a complex inner product space with orthonormal basis $\{e_1, e_2, e_3\}$. Let φ be an operator on V such that $\varphi(e_1) = e_1 + e_3$, $\varphi(e_2) = e_1 + e_2$, $\varphi(e_3) = e_2 + e_3$. Does there exist an orthonormal basis of V consisting of eigenvectors of φ? Motivate your answer.

Suggested solution:
The matrix for φ with respect to the orthonormal basis $\{e_1, e_2, e_3\}$ is

$$[\varphi] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

One easily checks that $[\varphi][\varphi]^* = [\varphi]^*[\varphi]$ so φ is a normal operator. By the complex spectral theorem there exist an orthonormal basis consisting of eigenvectors of φ, so the answer is yes.

Alternativly, one could diagonalize $[\varphi]$ and actually find an orthonormal basis.

7. Prove that all the eigenvalues of a self-adjoint operator on a complex inner product space are real numbers.

Suggested solution:
Let T be a self-adjoint operator on a complex inner product space, let λ be an eigenvalue and let v be a corresponding eigenvector. Then

$$\lambda\langle v, v\rangle = \langle Tv, v\rangle \ = \langle v, Tv\rangle \ = \langle v, \lambda v\rangle \ = \bar{\lambda}\langle v, v\rangle.$$

Since $\langle v, v\rangle$ is nonzero, we have $\lambda = \bar{\lambda}$ so λ is a real number.

8. a) Define addition and scalar multiplication on $\mathbb{R}^+$, the set of positive real numbers, so that it becomes a real vector space with additive identity 1.
b) Is it possible to define addition and scalar multiplication on $\mathbb{Q}$ so that it becomes a $\mathbb{Q}$-vector space of dimension 2? Motivate your answer.

Suggested solution:
Let $\varphi : V \to X$ be a bijection from a K-vector space V to any set X. Then we can transfer the vector space structure from V to X by defining the following operations on X:

$$x + y := \varphi(\varphi^{-1}(x) + \varphi^{-1}(y)) \qquad \lambda x := \varphi(\lambda\varphi^{-1}(x)) \qquad \forall\, x, y \in X;\ \lambda \in K$$

Then X is a K-vector space, and φ becomes a linear map by construction. All the vector space axioms follows from the corresponding axioms of V.

a) Using the bijection $t \mapsto e^t$ from $\mathbb{R}$ to $\mathbb{R}^+$, the equations above become

$$x + y := xy \qquad \lambda x := x^\lambda \qquad \forall\, x, y \in \mathbb{R}^+;\ \lambda \in \mathbb{R}$$

Note that the additive identity (the zero) is 1.

b) $\mathbb{Q}^2$ is a $\mathbb{Q}$-vector space of dimension 2 and $\mathbb{Q}^2$ is countable since $\mathbb{Q}$ is. This shows that there exists a bijection $\varphi : \mathbb{Q}^2 \to \mathbb{Q}$ and the construction above goes through. Since linear bijections between vector spaces maps bases to bases, the answer is yes.