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PROV I MATEMATIK
Linjär algebra III, 5hp
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Solutions

1. No such linear mapping exists since $(1,2,3) = 2\cdot(1,1,1) - (1,0,-1)$ but $2t+1 \neq 2\cdot(t^2+t+1) - (2t^2+t+2)$.

2. Assume $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$. Then, forming the inner product with the vector v_i, one obtains

$$c_i\|v_i\|^2 = \langle c_1v_1 + c_2v_2 + \cdots + c_nv_n, v_i\rangle = \langle 0, v_i\rangle = 0.$$

   We conclude that $c_i = 0$ for all i. Hence, the sequence is linearly independent.

3.
$$\chi_T(t) = \begin{vmatrix} t-3 & -1 & 1 \\ -2 & t-4 & 2 \\ -4 & -4 & t+2 \end{vmatrix} = (t-2)^2(t-1).$$

   The minimal polynomial $\phi_T(t)$ is a divisor of the characteristic polynomial and has the same roots. Hence, there are two possibilities – either $\phi_T(t) = (t-2)(t-1)$ or $\phi_T(t) = (t-2)^2(t-1)$. But since the matrix of $(T-2I)(T-I)$ is

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 4 & 4 & -4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & -1 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

   we conclude that

$$\phi_T(t) = (t-2)(t-1).$$

4. a) A *dual basis* $\phi_1, \phi_2, \dots, \phi_n$ of $e_1, e_2, \dots, e_n$ is a basis for V' (the space of all linear forms on V) such that $\phi_i(e_j) = \delta_{ij}$ for $i, j = 1, 2, \dots, n$.

   b) Let $\phi_1(x) = a_1x_1 + x_2$ and $\phi_2 = b_1x_1 + x_2$. Then ϕ_1, ϕ_2 will be the dual basis of e_1, e_2 if and only if

$$\begin{cases} \phi_1(e_1) = a_1 + a_2 = 1 \\ \phi_1(e_2) = a_1 + 2a_2 = 0 \end{cases} \quad \text{and} \quad \begin{cases} \phi_2(e_1) = b_1 + b_2 = 0 \\ \phi_2(e_2) = b_1 + 2b_2 = 1 \end{cases}$$

   The solutions are $(a_1, a_2) = (2, -1)$ and $(b_1, b_2) = (-1, 1)$.
   Answer: The dual basis consists of the two forms $2x_1 - x_2$ and $-x_1 + x_2$.

c) The wedge product $\phi \wedge \omega$ is an alternating 3-form on the vector space V and is defined by the equation:

$$\phi \wedge \omega(v_1, v_2, v_3) = \phi(v_1)\omega(v_2, v_3) - \phi(v_2)\omega(v_1, v_3) + \phi(v_3)\omega(v_1, v_2).$$

The alternating 3-form $\omega \wedge \phi$ is defined by the equation

$$\omega \wedge \phi(v_1, v_2, v_3) = \omega(v_1, v_2)\phi(v_3) - \omega(v_1, v_3)\phi(v_2) + \omega(v_2, v_3)\phi(v_1).$$

Thus, $\phi \wedge \omega = \omega \wedge \phi$. This is a special case of the general relation $\omega_1 \wedge \omega_2 = (-1)^{k_1 k_2}\omega_2 \wedge \omega_1$ for alternating k_1-forms ω_1 and k_2-forms ω_2.

5. T^* is a linear operator on $\mathbf{R}^2$. Thus, $T^*(y_1, y_2) = (ay_1 + by_2, cy_1 + dy_2)$ for suitable constants a, b, c, d. The defining relation $\langle Tx, y\rangle = \langle x, T^*y\rangle$ boils down to the equation

$$\begin{aligned}(2x_1 - x_2)y_1 + 2(2x_1 - x_2)y_2 + 2(3x_1 + 5x_x)y_1 + 5(3x_1 + 5x_2)y_2 \\ = x_1(ay_1 + by_2) + 2x_1(cy_1 + dy_2) + 2x_2(ay_1 + by_2) + 5x_2(cy_1 + dy_2).\end{aligned}$$

Both sides must have the same x_1y_1-coefficient. Hence,

$$a + 2c = 8.$$

Similarly, by comparing x_2y_1-, x_1y_2- and x_2y_2-coefficients:

$$2a + 5c = 9, \quad b + 2d = 19, \quad 2b + 5d = 23,$$

and by solving two systems of linear equations we obtain $a = 22$, $b = 49$, $c = -7$ and $d = -15$.

Thus,

$$T^*(y_1, y_2) = (22y_1 + 49y_2, -7y_1 - 15y_2).$$

A more transparent and general solution uses matrices and runs as follows: A general inner product on $\mathbf{R}^n$ can be written in the form

$$\langle x, y\rangle = x^t C y$$

where C is a positively definite matrix and where vectors x and y in $\mathbf{R}^n$ are written as column matrices. (In our case, $C = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$.) Now, let A be the matrix of the operator T, and B the matrix of the adjoint operator T^*. The coordinates of Tx and T^*y are then given by the column matrices Ax

and By, and the defining relation $\langle Tx, y\rangle = \langle x, T^*y\rangle$ can now be written as the matrix equation

$$(Ax)^t Cy = x^t CBy$$

that is, $x^t A^t Cy = x^t CBy$. This holds for all x and y if and only if

$$A^t C = CB,$$

and multiplying from the left by C^{-1} we obtain the general formula

$$B = C^{-1}A^t C,$$

which gives the relation between the matrices of T^* and T for a general inner product.

In our case, $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$. Hence,

$$B = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 22 & 49 \\ -7 & -15 \end{bmatrix}.$$

6.

$$\begin{aligned} v \in \mathcal{N}(T) \Leftrightarrow Tv = 0 \Leftrightarrow \langle v, T^*w\rangle = \langle Tv, w\rangle = 0 \quad \text{for all } w \in V \\ \Leftrightarrow v \perp \mathcal{V}(T^*) \Leftrightarrow v \in \mathcal{V}(T^*)^\perp. \end{aligned}$$

7. Since $\phi_{T_1}(t)$ has degree 8 and is a divisor of $\chi_{T_1}(t)$, which is a polynomial of degree 8, we first conclude that $\chi_{T_1}(t) = \phi_{T_1}(t)$. Since $\phi_{T_2}(t)$ has degree 6, is a divisor of $\chi_{T_2}(t)$, which is a polynomial of degree 7, and since the two polynomials have the same real roots, there is only one possibility left: $\chi_{T_2}(t) = (t-1)\phi_{T_2}(t) = (t-1)^3(t^2+1)^2$.

   Since T is the direct sum of the two operators T_1 and T_2, the characteristic polynomial $\chi_T(t)$ is the product of $\chi_{T_1}(t)$ and $\chi_{T_2}(t)$, and the minimal polynomial $\phi_T(t)$ is the least common multiple of $\phi_{T_1}(t)$ and $\phi_{T_2}(t)$.

   (To see this, choose a basis consisting of a basis from W_1 and a basis from W_2. The matrix of T has the form $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, where each submatrix A_i is the matrix of T_i. The determinant rules gives $\chi_T(t) = \det(tI - A) = \det(tI - A_1) \cdot \det(tI - A_2) = \chi_{T_1}(t)\chi_{T_2}(t)$. And for each polynomial $p(t)$, $p(A) = \begin{bmatrix} p(A_1) & 0 \\ 0 & p(A_2) \end{bmatrix}$. Therefore, $p(t)$ annihilates T (i. e. A) if and only if it is an annihilating polynomial for both T_1 and T_2, and the polynomial of least degree with this property is the least common multiple of the two minimal polynomials.)

Thus,

$$\chi_T(t) = (t-1)^4(t-2)(t^2+1)^5$$
$$\phi_T(t) = (t-1)^2(t-2)(t^2+1)^3$$

8. Start by computing the characteristic polynomials of S and T.

   a) A simple computation results in $\chi_S(t) = (t-1)^2(t-3)(t+1)$. The eigenspace corresponding to the double eigenvalue $t = 1$ is one-dimensional (and the minimal polynomial is

   $$\phi_S(t) = \chi_S(t) = (t-1)^2(t-3)(t+1)$$

   with an exponent $2 > 1$.) Thus, S is **not** diagonalizable. Since every normal operator is diagonalizable, we conclude that there is no inner product on V making S normal.

   b) The characteristic polynomial of T is $\chi_T(t) = (t+1)(t+2)(t-2)(t-4)$. Let e_1, e_2, e_3, e_4 be eigenvectors corresponding to the four distinct eigenvalues. The four eigenvectors form a basis for V. Now define an inner product on V by setting $\langle e_i, e_j \rangle = \delta_{ij}$. This makes V to an inner product space with e_1, e_2, e_3, e_4 as an ON-basis, and the matrix of T with respect to this ON-basis is diagonal with the eigenvalues as diagonal elements. Diagonal matrices are normal, so T is a normal operator, and T is even selfadjoint, since diagonal matrices with real elements are selfadjoint. Therefore, the answer to question b) is YES.