LINEAR ALGEBRA III
FINAL EXAM

Date 2019-08-19 **Course** 1MA026

No aid permitted, only writing utensils. Please write in English or Swedish. Motivate your answers. Each problem is worth 5 points. For grade 3, 4 and 5 you need 18, 25 and 32 points, respectively. Good luck!

1. For each of the following statements, determine if it is true or false. Justify your answer by a short proof, a counterexample, or by refering to a theorem.

a) If $m < n < \infty$ then there is no surjective linear map $\varphi : \mathbb{R}^m \to \mathbb{R}^n$.

b) If $V \otimes W \simeq \{0\}$ then either $V \simeq \{0\}$ or $W \simeq \{0\}$ (or both).

c) Every nonzero vector space contains infinitely many vectors.

d) If φ is a self-adjoint operator on a finite-dimensional complex inner product space, then all its eigenvalues are real.

2.

a) Give the general definition of an isomorphism of vector spaces.

b) Consider the real vector space $C^\infty(\mathbb{R})$ of smooth functions $f : \mathbb{R} \to \mathbb{R}$. Let V be the subspace with basis $\{\mathrm{e}^{nx} \mid n \in \mathbb{Z}_{>0}\}$. Let $D = \frac{\mathrm{d}}{\mathrm{d}x} : V \to V$ be the differential operator on V. Determine whether D is an isomorphism.

3. Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 6 & 9 \\ 0 & -4 & -6 \end{bmatrix}.$$

Find a Jordan matrix J and an invertible matrix S such that $A = SJS^{-1}$.

4. Consider the real vector space $\mathcal{P}_2$ of polynomials of degree at most 2 with coefficients in $\mathbb{R}$. Define a symmetric bilinear form $\beta : \mathcal{P}_2 \times \mathcal{P}_2 \to \mathbb{R}$ as

$$\beta(p, q) = p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

a) Determine whether β is an inner product on $\mathcal{P}_2$.

b) Find the orthogonal of the vector $x^2 + x \in \mathcal{P}_2$.

c) Find the orthogonal of the subspace $\mathcal{P}_0$ of constant polynomials.

5. Let φ be an operator on the complex vector space $\mathbb{C}^n$. Assume that $\varphi^n = i\varphi$.

a) Show that $\ker \varphi \neq \{0\}$.

b) Assume that $\dim\big(\ker \varphi\big) = n-2$. Find a Jordan normal form of φ.

6. Consider the complex vector space $\mathbb{C}^2$ equipped with the standard inner product. Let $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ be the linear map given by

$$\varphi(z_1, z_2) = (z_1 + \lambda z_2, iz_1 + z_2)$$

where $\lambda \in \mathbb{C}$ is some complex constant.

a) Find the adjoint of φ.

b) Determine for which values of the constant $\lambda \in \mathbb{C}$ the operator φ is **orthogonally** diagonalizable (i.e. for which $\lambda \in \mathbb{C}$ there exists an ON-basis of $\mathbb{C}^2$ consisting of eigenvectors of φ).

7. Consider the $\mathbb{R}$-vector space $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ of complex 2×2-matrices. Let

$$\mathcal{S} = \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid A = A^*\}$$

where $A^* = \bar{A}^t$ is the conjugate transpose of A.

a) Show that $\mathcal{S}$ is a subspace of the **real** vector space $\mathrm{Mat}_{2\times 2}(\mathbb{C})$.

b) Find a basis of the quotient $\mathrm{Mat}_{2\times 2}(\mathbb{C})/\mathcal{S}$.

8. Let V be a finite-dimensional complex vector space equipped with an inner product $\langle -, - \rangle$. Recall that a linear map $\varphi : V \to V$ is called an *isometry* with respect to $\langle -, - \rangle$ if it preserves the induced norm, i.e. if

$$\sqrt{\langle \varphi(v), \varphi(v) \rangle} = \sqrt{\langle v, v \rangle}$$

for every $v \in V$.

a) Show that a linear map $\varphi : V \to V$ is an isometry if and only if $\langle \varphi(u), \varphi(v) \rangle = \langle u, v \rangle$ for all $u, v \in V$.

b) Use a) to prove that φ is an isometry if and only if $\varphi^* \circ \varphi = \mathrm{id}_V$, where φ^* denotes the adjoint of φ.

(Hint: in a), consider $\langle \varphi(u+v), \varphi(u+v) \rangle$. In b) you may use that inner products are non-degenerate.)