LINEAR ALGEBRA III
FINAL EXAM

Date 2020-03-18 **Course** 1MA026

No aid permitted, only writing utensils. Please write in English or Swedish. Motivate your answers. Unless otherwise specified, each problem is worth 5 points. For grade 3, 4 and 5 you need 18, 25 and 32 points, respectively. Good luck!

1. For each of the following statements, determine if it is true or false. Justify your answer by a short proof, a counterexample, or by referring to a theorem.

a) Let $\varphi : V \to V$ be an endomorpism. If $v \in V$ is an eigenvector of φ, then v is an eigenvector of φ^2.

b) Any nonzero vector space contains infinitely many elements.

c) If $n < m < \infty$ then there is no injective linear map $\mathbb{R}^m \to \mathbb{R}^n$.

d) There is a finite-dimensional vector space V such that $V \otimes V \simeq V \oplus V$.

2. Consider the real vector space $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ of 2×2-matrices with entries from $\mathbb{R}$. Let $\mathcal{S}$ be the subset of matrices that commute with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, i.e.

$$\mathcal{S} = \left\{ A \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \mid A \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A \right\}.$$

Show that $\mathcal{S}$ is a subspace and find a basis of $\mathcal{S}$.

3. Let $\mathcal{C}^\infty([0,1])$ be the real vector space of smooth functions $[0,1] \to \mathbb{R}$, and consider the subspace $\mathcal{S}$ spanned by 1 and x. Together with

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$$

$\mathcal{C}^\infty([0,1])$ is an inner product space. Find the vector in $\mathcal{S}$ which is the closest to the vector e^x.

4. Find a Jordan matrix J and an invertible matrix S such that $S^{-1}AS = J$, where

$$A = \begin{bmatrix} -2 & 1 & 1 \\ -3 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

5. Let V be a vector space over some field K. Given two functionals α and β in the dual space of V, define their wedge product $\alpha \wedge \beta : V \times V \to K$ by

$$\alpha \wedge \beta(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u).$$

Show that $\alpha \wedge \beta$ is an alternating bilinear form on V.

6. The matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ -1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

has characteristic polynomial $\chi_A(x) = (x-2)^4$. Find a Jordan normal form of A.
Please note: you do ***not*** *have to find an invertible matrix S such that $S^{-1}AS = J$, only the Jordan matrix J.*

7. (10p) Consider $\mathbb{C}^2$ as a complex vector space, equipped with the standard inner product $\langle x, y \rangle = x_1\bar{y}_1 + x_2\bar{y}_2$. Let $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ be the operator which in the standard basis is given by the matrix

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

a) Find all eigenvalues and bases of the corresponding eigenspaces.

b) Using a), determine if φ is orthogonally diagonalizable.

c) State the Complex Spectral Theorem and use it to verify that your conclusion from b) is correct.

Please note: in b) you may not use the Spectral Theorem. If you did not do b), you can get full credit for c) by stating the Spectral Theorem and using it to determine if φ is orthogonally diagonalizable.)