Exam: Linear Algebra III

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The exam has total 8 questions in two pages. The credit for each question is written next to it. You can try questions in any order. You need to score 18 credits to get grade "3", 25 credits to get grade "4", 32 credits to get grade "5" in the exam including bonus points from assignments (if any).
In case of any doubts send an e-mail to any one of two above mentioned e-mails, I am available throughout your exam. Good luck!

For any positive integer n, $\mathbb{R}^n$ and $\mathbb{C}^n$ are equipped with their Euclidean inner products.

1. (a) For vector spaces V and W over a field F, give the definition of the null space of a linear map $T : V \to W$. [1 credit]
   (b) Let $\mathcal{P}_3(\mathbb{R})$ denote the vector space of all polynomials over $\mathbb{R}$ of degree at most 3. Find the null space of the differentiation map $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ given by $T(p) = p'$. [1 credit]
   (c) Prove or disprove if $T : \mathbb{C}^2 \to \mathbb{C}^2$ be a linear map such that 2 is an eigenvalue of T, then 5 is an eigenvalue of $p(T) = T^2 + I$. [1 credit]
   (d) Find a basis of $\mathbb{C}$ over $\mathbb{R}$ and also find a basis of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ over $\mathbb{R}$. [2 credit]

2. Does there exist a basis of $\mathbb{R}^3$ consisting of eigenvectors of $T \in \mathcal{L}(\mathbb{R}^3)$ defined by
$$T(x_1, x_2, x_3) = (4x_1 + x_2 + x_3, 2x_3, x_3)?$$
If yes, find such a basis. [5 credit]

3. (a) Find the adjoint T^* of $T : \mathbb{C}^3 \to \mathbb{C}^3$ defined by $T(x, y, z) = (ix + 2z, 3y + (i + 1)z, z)$. [2 credit]
   (b) Prove that there does not exist a self-adjoint operator $T \in \mathcal{L}(\mathbb{R}^3)$ such that $T(1, 2, 3) = (0, 0, 0)$ and $T(4, 5, 6) = (4, 5, 6)$. [2 credit]

4. Find an orthonormal basis of $\mathbb{R}^3$ by applying the Gram–Schmidt procedure to the basis
$$((1, 0, 0), (1, 1, 1), (1, 1, 2))$$
of $\mathbb{R}^3$. [5 credit]

5. Let $Q(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2$ be a quadratic form on $\mathbb{R}^2$. Find a corresponding symmetric matrix of Q and also determine the signature of Q. [5 credit]

6. Define $T \in \mathcal{L}(\mathbb{R}^3)$ by $T(x, y, z) = (2x, 3z, 2y)$. Find explicitly an isometry $S \in \mathcal{L}(\mathbb{R}^3)$ such that $T = S\sqrt{T^*T}$. [5 credit]

7. Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be a linear map whose matrix with respect to the standard basis of $\mathbb{C}^3$ is $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

   (a) Find the formula for T. [1 credit]

   (b) Find the eigenvalues of T. [1 credit]

   (c) Find the eigenspaces and the generalized eigenspaces corresponding to all the eigenvalues of T. [1 credit]

   (d) Find the Jordan form of T. [1 credit]

   (e) Find a Jordan basis for T. [2 credit]

8. Prove that $\mathcal{B}((x_1, x_2), (y_1, y_2)) = x_1y_2 - x_2y_1$ is a bilinear form on $\mathbb{R}^2$, where $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. Find the matrix of $\mathcal{B}$ with respect to basis $((1, 1), (1, 2))$. Is $\mathcal{B}$ an inner product also? Justify it. [5 credit]