Final Exam – Linear Algebra III, 1MA026

Uppsala University
Department of Mathematics

2022-03-17
duration of the exam: 5 hours

Please write your answers in **English.** There are 8 problems in this exam, and each one is worth 5 points. The grade limits are: 18 points for grade 3, 25 points for grade 4 and 32 points for grade 5. **You need to motivate every step in your solution to get the full score on each question.** Good luck!

In the exercises below, F is an arbitrary field (unless otherwise specified) and V is a vector space over F.

1. Let $V = \mathbb{C}^3$, with the standard hermitian inner product. Let $U = \operatorname{Span}(v_1, v_2)$, where
$$v_1 = (1, i, 2) \qquad v_2 = (1, 0, -i).$$
   (a) Find a basis of the orthogonal complement $U^\perp$.
   (b) Find a vector subspace $W \subset V$ such that $W \neq U^\perp$ and $W \oplus U = V$.

2. Consider the derivative operator $D \in \mathcal{L}(\mathcal{P}_n(F))$ defined by
$$D(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) = a_1 + 2 a_2 x + \ldots + n a_n x^{n-1}.$$
Note that this makes sense for every integer $n \geq 0$ and every field F.
   (a) Show that for every fixed $a \in F$, the vectors
$$v_0 = 1, v_1 = x - a, v_2 = (x-a)^2, \ldots, v_n = (x-a)^n$$
form a basis B_a of the $(n+1)$-dimensional vector space $\mathcal{P}_n(F)$.
   (b) Suppose that the characteristic of F, $\operatorname{char}(F)$, is either 0 or greater than n. Show that the dual basis $\varphi_0, \ldots, \varphi_n$ of $(\mathcal{P}_n(F))'$ that corresponds to the basis B_a is such that for every $0 \leq k \leq n$
$$\varphi_k(p(x)) = \frac{1}{k!} D^k(p(x))(a).$$
($D^k(p(x))(a)$ means: apply D to $p(x)$ k many times, then evaluate at $x = a$.)

3. Let V_1 and V_2 be the subsets of $\mathbb{R}^3$ given by
$$V_1 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x - y + z = 0\}$$
and
$$V_2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x - y + z = 2\}.$$
   (a) Is V_1 a vector subspace of $\mathbb{R}^3$? If yes, then find an orthonormal basis of V_1.
   (b) Is V_2 a vector subspace of $\mathbb{R}^3$? If yes, then find an orthonormal basis of V_2.
   (c) Find the point in V_1 that is the closest to the point $(1, 0, 0)$.
   (d) Find the point in V_2 that is the closest to the point $(1, 0, 0)$.

4. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}.$$

Find a Jordan matrix J and an invertible matrix C such that $J = C^{-1}AC$.

5. Let $V = \mathbb{R}^2$ and let $B : V \times V \to \mathbb{R}$ be the bilinear form given by

$$B\left((x_1, y_1), (x_2, y_2)\right) = 2\, x_1\, x_2 - 5\, y_1\, x_2 - x_1\, y_2 + 4\, y_1\, y_2.$$

(a) Is B symmetric? Is B alternating? Justify.

(b) Let e_1, e_2 be the standard basis of $\mathbb{R}^2$. Write the corresponding basis of $V \otimes V$.

(c) Is there a linear map $f : V \otimes V \to \mathbb{R}$ such that $f(v \otimes w) = B(v, w)$ for all $v, w \in \mathbb{R}^2$? Justify. If the answer is affirmative, then write the images under f of the vectors in a basis of $V \otimes V$ (for example, you can use the basis that you wrote in the previous answer).

6. Let $F = \mathbb{Z}/7\mathbb{Z}$ denote the field with 7 elements. Let $T \in \mathcal{L}(F^3, F^2)$ be given by $T(x, y, z) = (2x + 4y + 6z, x + y + z)$. Find a basis of the null space $\operatorname{null}(T)$ and compute the number of elements in the quotient space $F^3/\operatorname{null}(T)$.

7. Let $T \in \mathcal{L}(\mathbb{R}^3)$ be a linear operator whose matrix with respect to the standard basis of $\mathbb{R}^3$ is

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Let $S \in \mathcal{L}(\mathbb{C}^3)$ be a linear operator whose matrix with respect to the standard basis of $\mathbb{C}^3$ is also A.

In each of the following questions, even if the answer is affirmative, you are not required to find the orthonormal basis or the corresponding diagonal matrix.

(a) Consider the standard inner product in $\mathbb{R}^3$. Can you find an ordered orthonormal basis of $\mathbb{R}^3$ with respect to which T is diagonal? Explain why.

(b) Consider the standard inner product in $\mathbb{C}^3$. Can you find an ordered orthonormal basis of $\mathbb{C}^3$ with respect to which S is diagonal? Explain why.

8. Let V be a finite dimensional vector space over F. The minimal polynomial of $T \in \mathcal{L}(V)$ is the monic polynomial $m_T(x) \in \mathcal{P}(F)$ of smallest possible degree, for which $m_T(T) = 0$. You may use the following two facts without proving them:

   - Given a choice of ordered basis of V, let A be the matrix representing T with respect to that basis. Let $p(x) \in \mathcal{P}(F)$. Then, $p(T) = 0$ iff $p(A) = 0$.
   - If $p(x) \in \mathcal{P}(F)$ is any polynomial such that $p(T) = 0$, then $p(x) = m_T(x).q(x)$ for some $q(x) \in \mathcal{P}(F)$ (equivalently: m_T divides p).

   In this exercise, let $F = \mathbb{C}$.

   (a) Show that every eigenvalue of T is a root of $m_T(x)$.

   (b) Show that every root of $m_T(x)$ is an eigenvalue of T.

   (c) Let $(V, \langle ., . \rangle)$ be a finite dimensional complex inner product space and let $T \in \mathcal{L}(T)$ be a normal operator. Show that

   $$m_T(x) = (x - \lambda_1) \ldots (x - \lambda_m),$$

   where $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues of T.