Final Exam – Linear Algebra III, 1MA026

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duration of the exam: 5 hours

- Please write your answers in **English.**
- There are 8 problems in this exam, and each one is worth 5 points. The grade limits are: 18 points for grade 3, 25 points for grade 4 and 32 points for grade 5.
- For every question, you need to **justify every step** in your solution to get the full score.

Good luck!!

1. Let $U \subset \mathbb{R}^4$ be the vector subspace defined by

   $U = \{(w, x, y, z) \in \mathbb{R}^4 \,:\, 2w - x + 3y = 0, w + 2y - z = 0 \text{ and } w + x + 3y - 3z = 0\}.$

   Find an orthonormal basis of a vector subspace $V \subset \mathbb{R}^4$ such that $U \oplus V = \mathbb{R}^4$.

2. Let $V = C(0, 1)$ be the $\mathbb{C}$-vector space of continuous functions $f : [0, 1] \to \mathbb{C}$, with the inner product given by

$$\langle f, g\rangle = \int_0^1 f(x)\,\overline{g(x)}\;\mathrm{d}x.$$

   (a) Find an orthonormal basis of the vector subspace $U \subset V$ spanned by $f_1, f_2 \in V$ given by $f_1(x) = 1$ and $f_2(x) = x$.

   (b) Let $g \in V$ be given by $g(x) = e^x$. Find the vector $f \in U$ that minimizes $\|f - g\|$.

3. Let V be the vector space of polynomials of degree up to 6, with real coefficients. Let $T \in \mathcal{L}(V)$ be the linear operator given by $T(p(x)) = p'''(x)$ (the third derivative of the polynomial).

   (a) Find a basis of the quotient space $V/\ker(T)$, where $\ker(T)$ is the kernel of T.

   (b) Find a basis of $\mathrm{im}(T)$, where $\mathrm{im}(T)$ is the image of T.

   (c) Determine if the sum $\ker(T) + \mathrm{im}(T)$ is an internal direct sum.

4. Consider the matrix

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 4 & 3 \\ -4 & -2 & -2 \end{pmatrix}.$$

   You can use the fact that the characteristic polynomial of A is $(x-2)^3$ without proving it. Find a Jordan matrix J and an invertible matrix C such that $J = C^{-1}AC$.

5. Consider the quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ given by $q(x, y, z) = x^2 + 4xy + 4xz + 2z^2$.

   (a) Let $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the symmetric bilinear form such that $B(v, v) = q(v)$. Find the matrix A that represents B in the standard basis of $\mathbb{R}^3$,

   (b) Let $T \in \mathcal{L}(\mathbb{R}^3)$ be the linear operator represented by the matrix A above, in the standard basis of $\mathbb{R}^3$. Is there an orthonormal basis of $\mathbb{R}^3$ with respect to which the linear operator T is represented by a diagonal matrix D? If yes, find the matrix D (you don't need to find the basis). If no, justify.

6. (a) Write a matrix A whose characteristic polynomial is $p_A(x) = x^2(x+5)^3(x-7)^2$ and whose minimal polynomial is $m_A(x) = x(x+5)^2(x-7)^2$.

   (b) If B is a matrix such that $p_B(x) = p_A(x)$ and $m_B(x) = m_A(x)$, is there a matrix J that is a Jordan matrix for both A and B? Don't forget to justify your answer.

7. Let U and V be vector spaces over a field F. Each of the vector spaces could be finite or infinite dimensional. Use the universal property of the tensor product to define a linear map $T : U \otimes V \to V \otimes U$ that is an isomorphism of vector spaces. Don't forget to justify that the map is an isomorphism.

8. Let V be a vector space over a field F, with $\dim(V) = n < \infty$. An operator $N \in \mathcal{L}(V)$ is called nilpotent if $N^k = 0$ for some integer $k \geq 1$. Show that if N is nilpotent then $N^n = 0$.