5/1/ 2022

1a) $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \sim N_4 \left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{4} & 0 & 2 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 2 \end{pmatrix} \right)$

(a) $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2 (0, I_2) = N_2 (\mu_1, \Sigma_{11})$

(b) $\begin{pmatrix} X_3 \\ X_4 \end{pmatrix} \sim N_2 \left(\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \right) = N_2 (\mu_2, \Sigma_{22})$

$$\Sigma_{12} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{pmatrix}$$

$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \Big| \begin{pmatrix} X_3 \\ X_4 \end{pmatrix} \sim N \left(\mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (x_2 - \mu_2), \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \right)$

$$\Sigma_{22}^{-1} = \begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix}^{-1} = \frac{4}{15} \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{pmatrix}$$

$$\Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} = \begin{pmatrix} \frac{29}{30} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (x_2 - \mu_2) = \begin{pmatrix} \frac{1}{15} (2 x_3 - \frac{1}{2} x_4 \\ 0 \end{pmatrix}$$

(c) $E(X_3 | X_2) = EX_3 = 1$

(d) $E (2X_2 + 2 | (X_3, X_4) = 2 \underbrace{E(X_2 | X_3, X_4)}_{= EX_2 = 0} + 2 = 2$

2)

(a) measured $X_{ijk} = \begin{pmatrix} x_{1ij} \\ x_{2ij} \\ x_{3ij} \\ x_{4ij} \end{pmatrix}$

$i = 1 \dots n$ replications
$j = 1, 2, 3, 4$ coffee sorts
$k = 1, 2, 3$ used milk ~~...~~

$$X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}$$

$p \times 1$

$p = 4$ $\qquad \varepsilon_{ijk} \sim N_4(0, \Sigma)$ i.i.d.

Identification conditions:

$$\sum_i \alpha_i = \sum_j \beta_j = \sum_i \gamma_{ij} = \sum_j \gamma_{ij} = 0$$

Two-way with interaction

(b) $H_0: \gamma_{ij} = 0 \quad \forall i, j$

(c) Two way MANOVA without interactions.

$$X_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ij} \qquad \varepsilon_{ij} \sim N_4(0, \Sigma)$$
$$\sum_i \alpha_i = \sum_j \beta_j = 0 \qquad \text{i.i.d.}$$

(d) LRT

$$= \frac{\max\limits_{H_0} L(\mu, \Sigma)}{\max\limits_{\alpha,\beta,\gamma} L(\mu, \Sigma)}$$

α, β, γ

d)

Wilks Λ

$$\Lambda^* = \frac{SSP_{res}}{SSP_{int} + SSP_{res}}$$

Λ^* small $\rightsquigarrow$ rejecting

SSP_{res}

$$= \sum\sum\sum (x_{ijk} - \bar{x}_{ij})(\ )^T$$

SSP_{int}

$$= \sum_i \sum_j n(\bar{x}_{lk} - \bar{x}_{l\cdot} - \bar{x}_{\cdot k} + \bar{x})(\ )^T$$

e) p-value $= P_0(\Lambda^* < \Lambda^*_{obs})$

f) reject.
strong evidence against H_0

5.1.22

3)

(a) $X_1 \dots X_n$ coffee cream

$Z_1 \dots Z_n$ oat milk

$X_1, \dots, X_n$ i.i.d

$N_4(\mu_x, \Sigma)$ > incl.

$Z_1 \dots Z_n$ i.i.d $N_4(\mu_z, \Sigma)$

(e) $H_0 : \mu_x = \mu_z$

Hotelling T2-test

$$T = (\bar{X} - \bar{Z})^T S (\bar{X} - \bar{Z})$$

$n_1 = n_2 = n$

$$S_1 = \frac{1}{n-1} \underbrace{\Sigma (x_i - \bar{X})(\quad)^T}_{\sim W_p(n-1, \Sigma)} \sim W$$

$$S_2 = \frac{1}{n-1} \Sigma (\quad)(\quad)^T$$

$$S_{pool} = \frac{n_1 - 1}{n_1 + n_2 - 2} S_1 + \frac{n_2 - 1}{n_1 + n_2 - 2} S_2$$

$$= \frac{n-1}{2n-2} (S_1 + S_2) = \frac{1}{2}(S_1 + S_2)$$

$$2(n-1) S \sim W_p(2(n-1), \Sigma)$$

3)

a) $p\text{-Value} = P_0(T > t_{obs})$

c) $p\text{-value} = 0.67$

no difference

(d) $2(n-1)S_{pool} \sim W_p(2(n-1), \Sigma)$

(e) $S_1 \sim W_p(n-1, \Sigma)$

(4) CCA

$U = a^T X^{(1)}$
$V = b^T X^{(2)}$

(a) Set up of CCA

$\begin{pmatrix} X_{(1)} \\ X_{(2)} \end{pmatrix}$ $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$

$(U_{(1)}, V_{(1)})$ max: cor$(U_{(1)}, V_{(1)})$
$(U_{(2)}, V_{(2)})$ max cor(U, V) and orthogonal
$U_{(1)} \perp U_{(2)}$ $V_{(1)}, V_{(2)}$
cov$(U_1, V_2) = 0$
cov$(V_1, U_2) = 0$

Wanted relationship between $X_{(1)}, X_{(2)}$ study

$A = \Sigma_{11}^{-\frac{1}{2}} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11}^{-\frac{1}{2}} = (\quad)$

svd: eigen(A)

$\lambda_1 = 0.5457$ $\lambda_2 = 0.0009$

$e_1 = \begin{matrix} -0.8946 \\ -0.4467 \end{matrix}$ $e_2 = \begin{matrix} 0.446765 \\ -0.8946 \end{matrix}$

$B = \Sigma_{22}^{-\frac{1}{2}} \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-\frac{1}{2}} = (\quad)$

svd λ_1 , λ_2

$f_1 = \begin{pmatrix} 0.6161 \\ 0.78767 \end{pmatrix}$ $f_2 = \begin{pmatrix} -0.7876 \\ 0.6161 \end{pmatrix}$

(b) $\cancel{U_{(1)} = -0.896\, X_1 - 0.4467\, X_2}$

$\cancel{V_{(1)} = 0.6161\, X_3 + 0.7877\, X_4}$

$U_{(1)} = e_1^T \Sigma_{11}^{-\frac{1}{2}} X^{(1)}$, $V_{(1)} = f_1^T \Sigma_{22}^{-\frac{1}{2}} X^{(2)}$

(c) (i) no λ_2 is to small.

(ii) $p = 2$ that what only give a reprensdation of the data, no simplification

$$e1\left(\Sigma_{11}^{-\frac{1}{2}}\right) = \begin{pmatrix}-0.7561\\ -0.5767\end{pmatrix} = \begin{pmatrix}a\\ b\end{pmatrix}$$

$$f1\left(\Sigma_{22}^{-\frac{1}{2}}\right) = \begin{pmatrix}0.5629\\ 0.8806\end{pmatrix} = \begin{pmatrix}c\\ d\end{pmatrix}$$

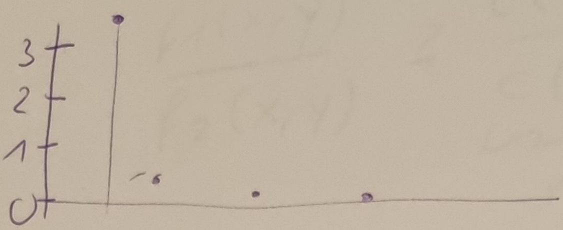
$$U_1 = aX_1 + bX_2$$

$$V_1 = cX_3 + dX_4$$

(5) PCA.

1.)

(a)



(b) 1 maybe 2

(c) $e_1^T X$, $e_2^T X$

(d) $\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_4 \end{pmatrix}$

6) a) ECM

$$\frac{f_1(x,y)}{f_2(x,y)} \geq \underbrace{\frac{c(1|2)}{c(2|1)}}_{=2} \underbrace{\frac{p_2}{p_1}}_{=1}$$

$$ECM = c(2,1)\,p(2|1) + c(1|2)\,p(1|2)\,p_2$$

a) $R_1 = \left\{ (x,y), \ \frac{f_1(x,y)}{f_2(x,y)} \geq 2 \right\}$

(c) estimate the optimal region.

OBS. median!

(7)

a) Similarity measure

$$s(x,y) = \frac{1}{d(x,y)} \quad \text{for } x \neq y$$

$d(x,y)$ distance > 0

$d(x,x) = 0$

$d(x,y) = d(y,x)$

$d(x,y) \leq d(x,z) + d(z,y)$

$s(x,x)$ max

$s(x,y) = s(y,x)$

$s(x,y) > 0$

(b) own measure first letter the same.

(c) similarity

$$\begin{array}{cccc} \text{Ge} & \text{Se} & \text{Fin} & \text{Est.} \\ 5 & & & \\ 3 & 5 & & \\ 0 & 0 & 5 & \\ 0 & 0 & 3 & 5 \end{array}$$

(d) multidem scalin

Ge Se clos.

Fin Est clos.

(8)

(a) $Y_i = \begin{pmatrix} \text{paper \& BL} \\ \text{paper \& SF} \end{pmatrix}$ breaking length, burst length

$X_i = \begin{pmatrix} AFL \\ LFF \\ LZST \end{pmatrix}$

$\underset{n\times 2}{Y} = \underset{n\times 2}{b} + \underset{n\times 3}{X}\ \underset{3\times 2}{B} + E$ $\qquad E_i \sim N_2(0, \Sigma)$ i.i.d.

$(E_{i1}\ E_{i2})$

(b) $\hat{B} =$

$$\underset{n\times 2}{\hat{b}} = \begin{pmatrix} \hat{b}_1 & \hat{b}_2 \end{pmatrix} \mathbb{1}_n \qquad \hat{b}_1 = -55,\ \hat{b}_2 = -36$$

$n\times 1 \quad 1\times 2$

$$B = \begin{pmatrix} -4 & -2 \\ 0 & 0 \\ 69 & 37 \end{pmatrix}$$

(c) $H_0\ B_{11} = 0$, $H_0\ B_{12} = 0$

$H_0\ B_{21} = 0$, $H_0\ B_{22} = 0$

$H_0\ B_{11} = B_{12} = 0$

$H_0\ B_{21} = B_{..} = 0$

$\Rightarrow$

(d) second!

(e) only

$Y_{1_i} = \beta_1 LFF_i + \beta_2 ZST_i + \varepsilon$

part of model2.