UPPSALA UNIVERSITET
Matematiska institutionen
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Prov i matematik
KandFy3, gylärare,STS, frist.
1MA032 Introduction to ODEs
2022-08-26

Writing time: 5 hrs. Allowed accessories: writing materials only. The credit for each problem is shown below. For the grades 3, 4 and 5, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be accompanied with explanatory text (in either English or Swedish). Maximum one solution per page.

1. (5 points)

a) Find the solution to the initial value problem

$$x^3y' + y = x^2y, \quad y(-1) = 1.$$

On which interval is the solution defined?

b) Give an example of a separable ODE where the solutions are implicitly given by

$$\sin(y) = e^x + x^2 + C.$$

Briefly explain why the ODE indeed is separable.

2. (5 points)

a) Find the general solution to

$$y'' - 4y' + 13y = 13x^2 + 5x + 2.$$

b) Give an example of a second order linear ODE with constant coefficients that has

$$y_1(x) = e^{2x}\sin(3x) + \frac{1}{x} \quad \text{and} \quad e^{2x}\cos(3x) + \frac{1}{x}$$

as solutions.

3. (5 points) There is a solution of the ODE

$$xy'' - (2x+1)y' + (x+1)y = 0, \quad x > 0$$

of the form

$$y_1(x) = e^{\alpha x}$$

for some choice of $\alpha \in \mathbb{R}$. Find all solutions to the ODE.

4. (5 points) Consider the differential equation

$$x^2y'' - \frac{3}{2}xy' + (x+1)y = 0.$$

a) Show that this equation has a regular singular point at $x = 0$.

b) Determine the indicial equation and its roots.

c) Find two series solutions for $x > 0$, one corresponding to each of the roots of the indicial equation. It's enough to give the first three terms and the recurrence relation for the coefficients.

Please turn the page!

5. (5 points) Consider the linear system

$$\begin{cases} x' &= -2x + 3y \\ y' &= 12x - 2y. \end{cases}$$

Find the general solution. Investigate the type and stability of the origin $(0,0)$ and sketch the phase portrait.

6. (5 points) Consider the inhomogeneous system

$$\begin{cases} x' &= y \\ y' &= -x + \dfrac{1}{\cos^3 t}. \end{cases}$$

Find the general solution.

7. (5 points) Consider the **Rayleigh equation**:

$$u'' - \mu\left(1 - \frac{1}{3}(u')^2\right)u' + u = 0,$$

where $\mu \in \mathbb{R}$.

a) Reduce the ODE to a system of first order equations.

b) Find all critical points and classify their type and stability for $\mu \neq 0$ (in the case of $\mu = \pm 2$ it is enough to determine the stability).

c) Are there (non-trivial) periodic trajectories contained strictly in the right half-plane?

8. (5 points) Consider the non-linear system

$$\begin{cases} x' &= -x^3 + 2xy^2 \\ y' &= -2x^2 y - y^3. \end{cases}$$

a) Find the critical points of the system.

b) Use Lyapunov's method, with a function of the form $V(x,y) = ax^k + cy^\ell$, to determine the type of stability of the origin $(0,0)$.

GOOD LUCK!

Svar till tentamen i 1MA032 Introduction to ODEs 2022-08-26