

Writing time: 5 hrs. Allowed accessories: writing materials only. The credit for each problem is shown below. For the grades 3, 4 and 5, one should obtain at least 18, 25 and 32 points, respectively. Solutions should be accompanied with explanatory text (in either English or Swedish). Maximum one solution per page.

1. (5 points)

a) Find the solution to the initial value problem

$$x^3 y' + y = x^2 y, \quad y(-1) = 1.$$

On which interval is the solution defined?

b) Give an example of a separable ODE where the solutions are implicitly given by

$$\sin(y) = e^x + x^2 + C.$$

Briefly explain why the ODE indeed is separable.

**Suggested solution:** By rewriting the equation as

$$x^3 y' + (1 - x^2)y = 0$$

we see that it is a first order linear equation. We can solve it using an integrating factor. Dividing by  $x^3$  gives us

$$y' + \left( \frac{1}{x^3} - \frac{1}{x} \right) y = 0$$

and the integrating factor is hence given by

$$\mu(x) = e^{\int \frac{1}{x^3} - \frac{1}{x}} = e^{-\frac{1}{2x^2} - \log|x|} = \frac{e^{-\frac{1}{2x^2}}}{|x|}.$$

Multiplying by the integrating factor we can write the equation as

$$\left( \frac{e^{-\frac{1}{2x^2}}}{|x|} y \right)' = 0.$$

Integrating and solving for  $y$  gives us

$$y = C|x|e^{\frac{1}{2x^2}}.$$

We are looking for a solution satisfying  $y(-1) = 1$ , plugging  $x = -1$  into the above we get

$$y(-1) = C|-1|e^{\frac{1}{2(-1)^2}} = Ce^{\frac{1}{2}}.$$

For this to equal 1 we must take  $C = e^{-\frac{1}{2}}$ , giving us the solution

$$y = |x|e^{\frac{1}{2x^2} - \frac{1}{2}}.$$

We can see that the function is well defined everywhere except at  $x = 0$ , since the initial value was taken at  $x = -1$  the solution is thus defined on the interval  $(-\infty, 0)$ . This solves a).

Implicitly differentiating the equation with respect to  $x$  gives us

$$\cos(y)y' = e^x + 2x,$$

which we can write as

$$y' = (e^x + 2x) \cdot \frac{1}{\cos(y)}.$$

Since this is in the form

$$y' = f(x)g(y)$$

the equation is separable, and by construction it has the required solution. This answers *b*).

**2.** (5 points)

a) Find the general solution to

$$y'' - 4y' + 13y = 13x^2 + 5x + 2.$$

b) Give an example of a second order linear ODE with constant coefficients that has

$$y_1(x) = e^{2x} \sin(3x) + \frac{1}{x} \quad \text{and} \quad e^{2x} \cos(3x) + \frac{1}{x}$$

as solutions.

**Suggested solution:** We start by solving the associated homogeneous equation

$$y'' - 4y' + 13y = 0.$$

The characteristic polynomial is

$$r^2 - 4r + 13,$$

with roots

$$r_{1,2} = 2 \pm 3i.$$

The general solution to the homogeneous equation is thus given by

$$y_h(x) = e^{2x}(C_1 \sin(3x) + C_2 \cos(3x)).$$

What remains is to find a particular solution. Since the right hand side is a second degree polynomial a natural guess for a particular solution would be

$$y_p(x) = Ax^2 + Bx + C.$$

Inserting this into the equation gives us

$$2A - 4(2Ax + B) + 13(Ax^2 + Bx + C) = 13x^2 + 5x + 2$$

which we can rewrite as

$$13Ax^2 + (-8A + 13B)x + 2A - 4B + 13C = 13x^2 + 5x + 2.$$

We get the solution  $A = 1$ ,  $B = 1$  and  $C = \frac{4}{13}$ , giving us

$$y_p(x) = x^2 + x + \frac{4}{13}.$$

The general solution to the equation is thus given by

$$y(x) = y_h(x) + y_p(x) = e^{2x}(C_1 \sin(3x) + C_2 \cos(3x)) + x^2 + x + \frac{4}{13}.$$

This answers a).

To answer b), we begin noticing that  $\frac{1}{x}$  is a common term for both  $y_1$  and  $y_2$ . If we would find an equation that has  $y_{h,1}(x) = e^{2x} \sin(3x)$  and  $y_{h,2}(x) = e^{2x} \cos(3x)$  as solutions to the associated homogeneous equation and  $y_p(x) = \frac{1}{x}$  as a particular solution then we would get that both

$$y_1 = y_{h,1} + y_p$$

and

$$y_2 = y_{h,2} + y_p$$

are solutions to this equation. Furthermore we can note that in the above problem we have precisely  $y_{h,1}(x) = e^{2x} \sin(3x)$  and  $y_{h,2}(x) = e^{2x} \cos(3x)$  as solutions to the associated homogeneous equation. Thus we only need to adjust the right hand side so that  $\frac{1}{x}$  becomes a particular solution, for this we can just insert  $y_p(x) = \frac{1}{x}$  into the left hand side of the equation and see what the right hand side must be, we get

$$y_p'' - 4y_p' + 13y_p = \frac{2}{x^3} + \frac{4}{x^2} + \frac{13}{x}.$$

So we have that  $y_p$  is a particular solution to the equation

$$y'' - 4y' + 13y = \frac{1}{2x^3} + \frac{4}{x^2} + \frac{13}{x}.$$

We already know that  $e^{2x} \sin(3x)$  and  $e^{2x} \cos(3x)$  are solutions to the associated homogeneous equation. It follows that  $y_1$  and  $y_2$  are solutions to the full equation.

**3.** (5 points) There is a solution of the ODE

$$xy'' - (2x + 1)y' + (x + 1)y = 0, \quad x > 0$$

of the form

$$y_1(x) = e^{\alpha x}$$

for some choice of  $\alpha \in \mathbb{R}$ . Find all solutions to the ODE.

**Suggested solution:** We start by finding  $\alpha$  such that  $y_1(x) = e^{\alpha x}$  is a solution. We have  $y_1'(x) = \alpha e^{\alpha x}$  and  $y_1''(x) = \alpha^2 e^{\alpha x}$ , inserting this into the equation gives us

$$x\alpha^2 e^{\alpha x} - (2x + 1)\alpha e^{\alpha x} + (x + 1)e^{\alpha x} = 0.$$

Division by  $e^{\alpha x}$  gives us

$$x\alpha^2 - (2x + 1)\alpha + (x + 1) = 0,$$

which we can rewrite as

$$(\alpha^2 - 2\alpha + 1)x - \alpha + 1 = 0.$$

For this to be equal to zero for all values of  $x$  we must have  $\alpha^2 - 2\alpha + 1 = 0$  and  $-\alpha + 1 = 0$ . The second equation gives us  $\alpha = 1$ , which we can see also solves the first equation. A solution to the equation is hence given by  $y_1(x) = e^x$ .

The next step is to find all solutions of the equation. Since it is a second order linear equation it has two linearly independent solutions, we have one,  $y_1$ , and want to find a second one. For this we use the method of reduction of order. We are thus looking for a solution of the form  $y_2 = y_1 u$ , inserting this into the equation we get

$$x(y_1 u)'' - (2x + 1)(y_1 u)' + (x + 1)y_1 u = 0.$$

Expanding the derivatives and combining the  $u$  terms we get

$$xy_1 u'' + (2xy_1' - (2x + 1)y_1)u' + (xy_1'' - (2x + 1)y_1' + (x + 1)y_1)u = 0.$$

Since  $y_1$  is a solution to the equation the last term is zero, if we also insert  $y_1 = e^x$  we are left with

$$xe^x u'' + (2xe^x - (2x + 1)e^x)u' = 0.$$

If we divide by  $e^x$  and let  $v = u'$  we get

$$xv' - v = 0.$$

This equation is both linear and separable, using either method a solution can be determined to be

$$v = x,$$

where we skip the constant of integration since we are only looking for one solution. Since  $u' = v$  (and  $x > 0$ ) we get  $u = \frac{1}{2}x^2$ , which gives us

$$y_2(x) = y_1(x)u(x) = e^x \frac{1}{2}x^2$$

as a second solution. From the theory of linear ODEs we now know that all solutions are on the form

$$y(x) = Ay_1(x) + By_2(x) = Ae^x + Bx^2e^x = (A + Bx^2)e^x$$

for some choice of constants  $A$  and  $B$ , where we have moved the  $\frac{1}{2}$  into the constant  $B$ .

4. (5 points) Consider the differential equation

$$x^2 y'' - \frac{3}{2}xy' + (x + 1)y = 0.$$

- Show that this equation has a regular singular point at  $x = 0$ .
- Determine the indicial equation and its roots.
- Find two series solutions for  $x > 0$ , one corresponding to each of the roots of the indicial equation. It's enough to give the first three terms and the recurrence relation for the coefficients.

**Suggested solution:** To show that  $x = 0$  is a regular singular point we first note that the equation can be written as

$$y'' + p(x)y' + q(x)y = 0$$

with

$$p(x) = -\frac{3}{2x} \quad \text{and} \quad q(x) = \frac{x + 1}{x^2}.$$

Both  $p(x)$  and  $q(x)$  are clearly singular at  $x = 0$ , so  $x = 0$  is a singular point. To see that it's regular we check that

$$xp(x) = -\frac{3}{2} \quad \text{and} \quad x^2q(x) = x + 1$$

are both analytic, which is clearly the case. This shows *a*).

The indicial equation is given by  $r^2 + (p_0 - 1)r + q_0 = 0$  where  $p_0$  is the value of  $xp(x)$  at  $x = 0$  and  $q_0$  is the value of  $x^2q(x)$  at  $x = 0$ . We get

$$p_0 = -\frac{3}{2} \quad \text{and} \quad q_0 = 1.$$

Giving us the indicial equation

$$r^2 - \frac{5}{2}r + 1 = 0$$

which has the roots

$$r_1 = \frac{1}{2} \quad \text{and} \quad r_2 = 2.$$

This answers *b*).

Finally we answer c). According to the method of Frobenius we are looking for series solutions of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

where  $r$  is one of the roots of the indicial equation.

We have

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r}, \\ y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n-1+r}, \\ y'' &= \sum_{n=0}^{\infty} (n+r)(n-1+r) a_n x^{n-2+r}. \end{aligned}$$

Plugging this into the equation we have

$$\sum_{n=0}^{\infty} (n+r)(n-1+r) a_n x^{n+r} - \frac{3}{2} \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+1+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Division by  $x^r$  and adjusting the indices gives us

$$\sum_{n=0}^{\infty} (n+r)(n-1+r) a_n x^n - \frac{3}{2} \sum_{n=0}^{\infty} (n+r) a_n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

If we take out the terms for  $n=0$  and put the remaining one in one sum we get

$$\left( r^2 - \frac{5}{2}r + 1 \right) + \sum_{n=1}^{\infty} \left( \left( (n+r)^2 - \frac{5}{2}(n+r) + 1 \right) a_n + a_{n-1} \right) x^n = 0$$

When  $r$  is a root of the indicial equation the first term is zero, for the sum to equal zero we must (by the identity principle) have

$$\left( (n+r)^2 - \frac{5}{2}(n+r) + 1 \right) a_n + a_{n-1} = 0 \iff a_n = -\frac{a_{n-1}}{(n+r)^2 - \frac{5}{2}(n+r) + 1}$$

for  $n = 1, 2, \dots$ . Since we are only looking for a particular solution we can take  $a_0 = 1$ .

For  $r_1 = \frac{1}{2}$  we get

$$a_n = -\frac{a_{n-1}}{n^2 - \frac{3}{2}n}$$

and the three first terms are given by

$$\begin{aligned} a_0 &= 1, \\ a_1 &= 2, \\ a_2 &= -2. \end{aligned}$$

For  $r_2 = 2$  we get

$$a_n = -\frac{a_{n-1}}{n^2 + \frac{3}{2}n}$$

and the three first terms are given by

$$\begin{aligned} a_0 &= 1, \\ a_1 &= -\frac{2}{5}, \\ a_2 &= \frac{2}{35}. \end{aligned}$$

5. (5 points) Consider the linear system

$$\begin{cases} x' &= -2x + 3y \\ y' &= 12x - 2y. \end{cases}$$

Find the general solution. Investigate the type and stability of the origin  $(0,0)$  and sketch the phase portrait.

**Suggested solution:** The system is of the form  $\dot{X} = AX$ , where

$$A = \begin{pmatrix} -2 & 3 \\ 12 & -2 \end{pmatrix}.$$

It is not difficult to see that the eigenvalues are  $\lambda_+ = 4$  and  $\lambda_- = -8$ , with associated eigenvectors

$$K_+ = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad K_- = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Hence, writing  $X_{\pm} = e^{\lambda_{\pm}t}K_{\pm}$ , the general solution is

$$X = C_+X_+ + C_-X_- = C_+e^{4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_-e^{-8t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

We see that as  $t \rightarrow \infty$  there is  $X \approx C_+X_+$  and for  $t \rightarrow -\infty$  we have  $X \approx C_-X_-$ . We depict the situation in Figure 1.

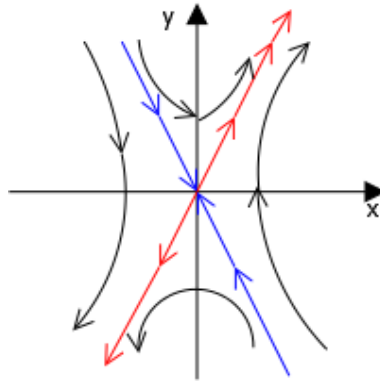


Figure 1: Phase portrait with the  $X_-$ -line in blue and the  $X_+$ -line in red.

6. (5 points) Consider the inhomogeneous system

$$\begin{cases} x' &= y \\ y' &= -x + \frac{1}{\cos^3 t}. \end{cases}$$

Find the general solution.

**Suggested solution:** We write the system in the form  $\dot{X} = AX + F$ , where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 \\ \frac{1}{\cos^3 t} \end{pmatrix}.$$

It is straightforward to find the eigenvalues of  $A$  to be  $\lambda_{\pm} = \pm i$ . The eigenvector  $K_+$  associated to  $i$  is

$$K_+ = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

and so the general homogeneous solution is  $X_h = e^{it}K_+$ , or

$$X = e^{it}K_+ = \dots = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Reading of real and imaginary parts as done in the lectures yields the homogeneous solution:

$$X_h = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

To find the particular solution  $X_p$  we form the fundamental matrix  $\Phi$  and find its inverse  $\Phi^{-1}$  (as done in the Lectures):

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \text{and} \quad \Phi^{-1}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

As in the lectures we now use the formula for  $X_p$ :

$$\begin{aligned} X_p &= \Phi(t) \int \Phi^{-1}(s)F(s)ds \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \int \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\cos^3 s} \end{pmatrix} ds \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \int \begin{pmatrix} -\frac{\sin s}{\cos^3 s} \\ \frac{1}{\cos^2 s} \end{pmatrix} ds \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \frac{1}{\cos^2 t} \\ \tan t \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} \frac{1}{\cos^2 t} + \sin t \cdot \tan t \\ \frac{1}{\sin t} \\ -\frac{1}{2} \frac{1}{\cos^2 t} + \cos t \cdot \tan t \end{pmatrix}. \end{aligned}$$

Hence, the total solution is

$$\begin{aligned} X &= X_h + X_p \\ &= C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \frac{1}{\cos^2 t} + \sin t \cdot \tan t \\ \frac{1}{\sin t} \\ -\frac{1}{2} \frac{1}{\cos^2 t} + \cos t \cdot \tan t \end{pmatrix}. \end{aligned}$$

7. (5 points) Consider the **Rayleigh equation**:

$$u'' - \mu \left( 1 - \frac{1}{3}(u')^2 \right) u' + u = 0,$$

where  $\mu \in \mathbb{R}$ .

- Reduce the ODE to a system of first order equations.
- Find all critical points and classify their type and stability for  $\mu \neq 0$  (in the case of  $\mu = \pm 2$  it is enough to determine the stability).
- Are there (non-trivial) periodic trajectories contained strictly in the right half-plane?

**Suggested solution:** We invoke the variables  $x = u$  and  $y = u'$  and obtain the following equation system:

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= \mu \left( 1 - \frac{1}{3}y^2 \right) y - x. \end{cases}$$

It is not difficult to see that the only critical point is the origin and the system is locally linear since

$$\lim_{(x,y) \rightarrow (0,0)} \frac{-\frac{\mu}{3}y^3}{\sqrt{x^2 + y^2}} = 0,$$

which can be seen by changing to polar coordinates. That the system is locally linear also follows since the right hand sides in the equation system are polynomials, which are  $C^2$ -functions. To investigate type and stability we analyze the linear part of the system  $\dot{X} = AX + F$ , where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}.$$

Via the standard calculation we obtain the eigenvalues of  $A$ :

$$\lambda_{\pm} = \frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 - 4}.$$

We see that the eigenvalues are real and distinct when  $|\mu| > 2$ , real but repeated when  $|\mu| = 2$  and complex when  $|\mu| < 2$ . In the case of  $|\mu| > 2$ , we observe that if  $\mu > 0$  both eigenvalues are positive (since  $\mu > \sqrt{\mu^2 - 4}$  for  $\mu > 0$ ) so that the system is a **unstable node**. If  $|\mu| > 2$  but  $\mu < 0$ , then both eigenvalues are negative (since  $\mu < \sqrt{\mu^2 - 4}$  for  $\mu < 0$ ) and the system is a **stable node**. If  $\mu = \pm 2$  we can only assert that the system is **stable** if  $\mu = -2$  and **unstable** if  $\mu = 2$ . Finally when  $|\mu| < 2$  we have the complex eigenvalues which have positive real part when  $\mu > 0$  and negative real part when  $\mu < 0$  corresponding to **unstable-** and **stable spirals**, respectively.

Finally, since the only critical point is the origin we can assert that there are no non-trivial trajectories in the right half-plane, as any such trajectory must enclose a critical point.

8. (5 points) Consider the non-linear system

$$\begin{cases} x' &= -x^3 + 2xy^2 \\ y' &= -2x^2y - y^3. \end{cases}$$

- Find the critical points of the system.
- Use Lyapunov's method, with a function of the form  $V(x, y) = ax^k + cy^\ell$ , to determine the type of stability of the origin  $(0, 0)$ .

**Suggested solution:** To find critical points we first observe that  $\dot{x} = 0$  implies  $x(2y^2 - x^2) = 0$  and  $\dot{y} = 0$  implies  $y(2x^2 + y^2) = 0$ . The latter equation implies  $y = 0$  so that the former implies  $x = 0$  and so the origin is the only critical point. To analyze its stability, we make the ansatz  $V(x, y) = ax^k + cy^\ell$  and insert the system:

$$\begin{aligned} \frac{dV}{dt} &= akx^{k-1}x' + cly^{\ell-1}y' \\ &= akx^{k-1}(-x^3 + 2xy^2) + cly^{\ell-1}(-2x^2y - y^3) \\ &= -akx^{k+2} + 2akx^ky^2 - 2cly^{\ell+2} - cly^{\ell+2}. \end{aligned}$$

Choosing  $k = \ell = 2$  and  $a = c = 1$  we get

$$\frac{dV}{dt} = -2x^4 - 2y^4 < 0,$$

when  $(x, y) \neq 0$ . Hence the origin is **asymptotically stable**.

**GOOD LUCK!**

