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1MA032 Ordinary differential equations I

2023–08–14

Allowed aids: writing materials. Each problem has a maximum credit of 5 points. For the grades 3, 4 and 5, respectively, one should obtain at least 18, 25 and 32 point, respectively. Solutions must be accompanied with explanatory text.

1. **a)** Find the solution to the initial value problem

$$xy' = y^2, \quad y(1) = 2.$$

b) Give an example of an exact ODE where the solutions are implicitly given by

$$\sin(y) + x^2 = C$$

Briefly explain why the ODE is exact.

(5 points)

2. The ODE

$$(x-1)y'' - xy' + y = 0, \ x > 1$$

has $y_1(x) = e^x$ as one solution. Find a solution to the ODE satisfying the initial conditions $y(2) = e^2 + 4$ and $y'(2) = e^2 + 2$.

The original exam had a wrong sign which made solving for v in the solution harder than intended.

(5 points)

3. Give an example of a second order linear ODE with constant coefficients that has

$$y_1(x) = (1 + x + x^2)e^{2x} \text{ and } y_2(x) = (2 + 2x + x^2)e^{2x}$$

as solutions.

(5 points)

4. Consider the differential equation

$$x^2y'' + \left(2x + \frac{7}{2}\right)xy' + \frac{3}{2}y = 0$$

a) Show that this equation has a regular singular point at $x = 0$.

b) Determine the indicial equation and its roots.

c) Find two series solutions for $x > 0$, one corresponding to each of the roots of the indicial equation. It's enough to give the first three terms and the recurrence relation for the coefficients. *Hint: If the first coefficients for the two series solutions are a_0 and b_0, then the fourth coefficients are $a_3 = -\frac{4}{15}a_0$ and $b_3 = 0$ respectively.*

(5 points)

5. Let

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \text{och} \quad A = \begin{pmatrix} 10 & -8 \\ 8 & -10 \end{pmatrix}.$$

a) Find the general solution to $X' = AX$.

b) Determine the type an stability of the critical point at the origin and sketch the phase portrait.

c) Find a solution satisfying

$$X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(5 points)

6. Determine the general solution to the system

$$\begin{cases} x' &= 3y + e^{2t} + 5 \\ y' &= x + 2y + e^{2t} - 3 \end{cases}.$$

(5 points)

7. Consider the ODE

$$\frac{d^2u}{dt^2} + c\frac{du}{dt} + u(1-u)(u-a) = 0$$

with $0 < a < 1$ and $c > 0$.

a) Rewrite the equation as a system of first order equations.

b) Determine the critical points of the system.

c) Determine the type and stability of the critical points, assuming $c^2 < 4a - 4a^2$.

(5 points)

8. Consider the system

$$\begin{cases} x' &= x^4 + 8y^5 \\ y' &= 2x^2 - 2y^3 \end{cases}.$$

a) Find all critical points and show that they are isolated.

b) Determine the stability of the origin. *Hint: Search for a Lyapunov function of the form* $V(x,y) = ax^k + by^l$.

(5 points)

Solutions to exam in 1MA032 Ordinary differential equations I 2023–08–14

Solution to problem 1. **a)** The equation is separable and for $x \neq 0$ and $y \neq 0$ we can write it as

$$\frac{1}{y^2}dy = \frac{1}{x}dx.$$

Integrating both sides we get

$$-\frac{1}{y} = \log|x| + C.$$

Solving for y gives us

$$y = -\frac{1}{\log|x| + C}.$$

The initial value $y(1) = 2$ gives the equation

$$-\frac{1}{\log|1| + C} = 2$$

from which we get $C = -\frac{1}{2}$. The final solution is hence

$$y = -\frac{1}{\log x - 1/2}.$$

Where we have removed the absolute value since $x > 0$ around the initial point.

b) The solutions to an exact differential equation

$$M(x,y) + N(x,y)y' = 0$$

are implicitly given by $F(x,y) = C$ where

$$\frac{\partial F}{\partial x}(x,y) = M(x,y) \text{ and } \frac{\partial F}{\partial y}(x,y) = N(x,y).$$

If we want the solutions to be $\sin(y) + x^2 = C$ we can hence take

$$F(x,y) = \sin(y) + x^2.$$

This would give us

$$M(x,y) = \frac{\partial F}{\partial x}(x,y) = 2x \text{ and } N(x,y) = \frac{\partial F}{\partial y}(x,y) = \cos(y).$$

We hence get the equation

$$2x + \cos(y)y' = 0,$$

which is exact by construction.

Solution to problem 2. Since we have one solution to the ODE we can find another using the method of reduction of order. We then make the ansatz $y_2(x) = y_1(x)u(x)$. Inserting this ansatz into the equation gives us

$$(x-1)(y_1''u + 2y_1'u' + y_1u'') - x(y_1'u + y_1u') + y_1u = 0.$$

Reordering it as

$$((x-1)y_1'' - xy_1' + y_1)u + (x-1)y_1u'' + (2(x-1)y_1' - xy_1)u' = 0$$

we see that the first term disappears since y_1 is a solution to the equation. Since $y_1 = y_1' = e^x$ we then get the equation

$$(x-1)e^xu'' + (x-2)e^xu' = 0.$$

Dividing by e^x and letting $v = u'$ we get the first order ODE

$$(x-1)v' + (x-2)v = 0.$$

The first order equation is separable and we can write it as

$$\frac{1}{v} dv = -\frac{x-2}{x-1} dx.$$

Integrating both sides we get

$$\log|v| = -x + \log|x-1| + C.$$

Taking the exponential and using that $x > 1$ gives us

$$v = \pm e^{-x+\log|x-1|+C} = \pm e^C (x-1)e^{-x}.$$

We can note that $\pm e^C$ is an arbitrary non-zero constant, since we get a solution also for $v = 0$ we can replace it with the arbitrary constant D to get

$$v = D(x-1)e^{-x}.$$

We now get

$$u = \int v\ dx = D\int (x-1)e^{-x}\ dx = -Dxe^{-x} + E.$$

This gives us

$$y_2(x) = y_1(x)u(x) = e^x(-Dxe^{-x} + E) = -Dx + Ee^x.$$

Taking $D = -1$ and $E = 0$ we get the solution $y_2(x) = x$.

The general solution to the equation is given by

$$y(x) = C_1 y_1(x) + C_2 y_2(x) = C_1 e^x + C_2 x,$$

with $y'(x) = C_1 e^x + C_2$. To satisfy the initial condition we get the system of equations

$$\begin{aligned} C_1 e^2 + 2C_2 &= e^2 + 4, \\ C_1 e^2 + C_2 &= e^2 + 2. \end{aligned}$$

We see that we get a solution by taking $C_1 = 1$ and $C_2 = 2$, giving us

$$y(x) = e^x + 2x.$$

Solution to problem 3. The solution of a second order linear ODE is of the form

$$y(x) = C_1 y_{h,1}(x) + C_2 y_{h,2}(x) + y_p(x)$$

where $C_1 y_{h,1}(x) + C_2 y_{h,2}(x)$ solves the associated homogeneous equation and $y_p(x)$ is a particular solution. We note that if

$$y_{h,1}(x) = e^{2x},\ y_{h,2}(x) = xe^{2x} \text{ and } y_p(x) = x^2 e^{2x}$$

then we can write $y_1(x)$ and $y_2(x)$ as

$$y_1(x) = y_{h,1}(x) + y_{h,2}(x) + y_p(x) \text{ and } y_2(x) = 2y_{h,1}(x) + 2y_{h,2}(x) + y_p(x).$$

We begin by finding a homogeneous second order linear ODE with constant coefficients that has $y_{h,1}(x) = e^{2x}$ and $y_{h,2}(x) = xe^{2x}$ as solutions. For e^{2x} to be a solution one root of the characteristic equation must be 2, for xe^{2x} to be a solution 2 must be a double root to the characteristic equation. We hence get that the characteristic equation is given by $(r-2)^2 = 0 \iff r^2 - 4r + 4$, giving us the equation

$$y'' - 4y' + 4y = 0.$$

The next step is to determine $f(x)$ such that the equation

$$y'' - 4y' + 4y = f(x)$$

has $y_p(x) = x^2e^{2x}$ as a particular solution. Inserting y_p into the left hand side we get

$$(2e^{2x} + 8xe^{2x} + 4x^2e^{2x}) - 4(2xe^{2x} + 2x^2e^{2x}) + 4x^2e^{2x} = f(x).$$

Simplifying we get

$$f(x) = 2e^{2x}.$$

Summarising we have that the equation

$$y'' - 4y' + 4y = 2e^{2x}$$

has $y_{h,1}(x) = e^{2x}$ and $y_{h,2}(x) = xe^{2x}$ as solutions to the associated homogeneous equation and $y_p(x) = x^2e^{2x}$ as a particular solution. Since

$$y_1(x) = y_{h,1}(x) + y_{h,2}(x) + y_p(x) \text{ and } y_2(x) = 2y_{h,1}(x) + 2y_{h,2}(x) + y_p(x).$$

both y_1 and y_2 are solutions to the equation.

Solution to problem 4. **a)** To show that $x = 0$ is a regular singular point we first note that the equation can be written as

$$y'' + p(x)y' + q(x)y = 0$$

with

$$p(x) = 2 + \frac{7}{2x} \quad \text{and} \quad q(x) = \frac{3}{2x^2}$$

Both $p(x)$ and $q(x)$ are clearly singular at $x = 0$, so $x = 0$ is a singular point. To see that it's regular we check that

$$xp(x) = 2x + \frac{7}{2} \quad \text{and} \quad x^2q(x) = \frac{3}{2}$$

are both analytic, which is clearly the case.

b) The indicial equation is given by $r^2 + (p_0 - 1)r + q_0 = 0$ where p_0 is the value of $xp(x)$ at $x = 0$ and q_0 is the value of $x^2q(x)$ at $x = 0$. We get

$$p_0 = \frac{7}{2} \quad \text{and} \quad q_0 = \frac{3}{2}.$$

Giving us the indicial equation

$$r^2 + \frac{5}{2}r + \frac{3}{2} = 0$$

which has the roots

$$r_1 = -\frac{3}{2} \quad \text{and} \quad r_2 = -1.$$

c) According to the method of Frobenius we are looking for series solutions of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

where r is one of the roots of the indicial equation.

We have

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r}, \\ y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n-1+r}, \\ y'' &= \sum_{n=0}^{\infty} (n+r)(n-1+r) a_n x^{n-2+r}. \end{aligned}$$

Plugging this into the equation we have

$$\sum_{n=0}^{\infty}(n+r)(n-1+r)a_nx^{n+r}+2\sum_{n=0}^{\infty}(n+r)a_nx^{n+1+r}+\frac{7}{2}\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}+\frac{3}{2}\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

Division by x^r and adjusting the indices gives us

$$\sum_{n=0}^{\infty}(n+r)(n-1+r)a_nx^{n}+2\sum_{n=1}^{\infty}(n-1+r)a_{n-1}x^{n}+\frac{7}{2}\sum_{n=0}^{\infty}(n+r)a_nx^{n}+\frac{3}{2}\sum_{n=0}^{\infty}a_nx^{n}=0$$

If we take out the terms for $n=0$ and put the remaining one in one sum we get

$$\left(r^2+\frac{5}{2}r+\frac{3}{2}\right)+\sum_{n=1}^{\infty}\left(\left((n+r)^2+\frac{5}{2}(n+r)+\frac{3}{2}\right)a_n+2(n-1+r)a_{n-1}\right)x^n=0$$

When r is a root of the indicial equation the first term is zero, for the sum to equal zero we must (by the identity principle) have

$$\left((n+r)^2+\frac{5}{2}(n+r)+\frac{3}{2}\right)a_n+2(n-1+r)a_{n-1}=0 \iff a_n=-2\frac{n-1+r}{(n+r)^2+\frac{5}{2}(n+r)+\frac{3}{2}}a_{n-1}$$

for $n=1,2,\ldots$. Since we are only looking for a particular solution we can take $a_0=1$.

For $r_1=-\dfrac{3}{2}$ we get

$$a_n=-2\frac{n-5/2}{(n-3/2)^2+\frac{5}{2}(n-3/2)+\frac{3}{2}}a_{n-1}$$

and the first four terms are given by

$$\begin{aligned}
a_0&=1,\\
a_1&=-2\frac{1-5/2}{(1-3/2)^2+\frac{5}{2}(1-3/2)+\frac{3}{2}}=6,\\
a_2&=-2\frac{2-5/2}{(2-3/2)^2+\frac{5}{2}(2-3/2)+\frac{3}{2}}a_1=2,\\
a_3&=-2\frac{3-5/2}{(3-3/2)^2+\frac{5}{2}(3-3/2)+\frac{3}{2}}a_2=-\frac{4}{15}.
\end{aligned}$$

For $r_2=-1$ we get

$$a_n=-2\frac{n-2}{(n-1)^2+\frac{5}{2}(n-1)+\frac{3}{2}}a_{n-1}$$

and the first four terms are given by

$$\begin{aligned}
a_0&=1,\\
a_1&=-2\frac{1-2}{(1-1)^2+\frac{5}{2}(1-1)+\frac{3}{2}}=\frac{4}{3},\\
a_2&=-2\frac{2-2}{(2-1)^2+\frac{5}{2}(2-1)+\frac{3}{2}}a_1=0,\\
a_3&=0.
\end{aligned}$$

Solution to problem 5. **a)** We start by computing the eigenvalues of the matrix A. We have

$$\det(A-\lambda I)=(10-\lambda)(-10-\lambda)+64=\lambda^2-36,$$

with the roots $\lambda_{1,2}=\pm 6$. The next step is to compute the eigenvectors. For $\lambda_1=-6$ we get the system

$$\begin{pmatrix}16 & -8\\ 8 & -4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

One solution is given by $x = 1$ and $y = 2$, giving us the eigenvector

$$K_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = 6$ we in the same way get the eigenvector

$$K_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Since we have two distinct real eigenvalues the general solution is given by

$$X(t) = C_1 K_1 e^{\lambda_1 t} + C_2 K_2 e^{\lambda_2 t}.$$

b) Since we have one negative and one positive eigenvalue we get that the origin is a saddle point, which is unstable.

c) For the general solution we get

$$X(0) = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

To have

$$X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

we must therefore solve the system

$$\begin{cases} C_1 + 2C_2 & = 1 \\ 2C_1 + C_2 & = 2 \end{cases}.$$

Which we can see has the solution $C_1 = 1$, $C_2 = 0$. The solution is therefore given by

$$X(t) = K_1 e^{\lambda_1 t}.$$

Solution to problem 6. We start by solving the associated homogeneous equation, given by $X' = AX$ with

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}.$$

The eigenvalues of A are given by the roots of

$$\det(A - \lambda I) = -\lambda(2 - \lambda) - 3 = \lambda^2 - 2\lambda - 3,$$

which are $\lambda_{1,2} = 1 \pm 2$. For the two eigenvectors we get the system of equations

$$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} K_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} K_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One choice of solutions is given by

$$K_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad \text{och} \quad K_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution to the associated homogeneous system is hence

$$X_h(x) = C_1 e^{-t} \begin{pmatrix} 3 \\ -1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The next step is to find a particular solution. We use the variation of parameter method. We then need the fundamental matrix, which in this case is given by

$$\Phi = \begin{pmatrix} 3e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix}.$$

By the method of variation of parameter a solution is then given by

$$X_p = \Phi(t) \int \Phi^{-1}(t) F(t) \, dt.$$

where $F(t)$ is the inhomogeneous part of the equation. We have $\det \Phi = 3e^{-t}e^{3t} + e^{-t}e^{3t} = 4e^{2t}$, so

$$\Phi^{-1}(t) = \frac{1}{\det \Phi} \begin{pmatrix} e^{3t} & -e^{3t} \\ e^{-t} & 3e^{-t} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} e^{t} & -e^{t} \\ e^{-3t} & 3e^{-3t} \end{pmatrix}.$$

This gives us

$$X_p = \Phi \int \frac{1}{4} \begin{pmatrix} e^{t} & -e^{t} \\ e^{-3t} & 3e^{-3t} \end{pmatrix} \begin{pmatrix} e^{2t} + 5 \\ e^{2t} - 3 \end{pmatrix} \, dt = \Phi \int \begin{pmatrix} 2e^{t} \\ e^{-t} - e^{-3t} \end{pmatrix} \, dt = \Phi \begin{pmatrix} 2e^{t} \\ -e^{-t} + \frac{1}{3}e^{-3t} \end{pmatrix},$$

where we have skipped the constants of integration since we only need one solution. In the end we get

$$X_p = \begin{pmatrix} -e^{2t} + \frac{19}{3} \\ -e^{2t} - \frac{5}{3} \end{pmatrix}.$$

So the general solution is given by

$$X = X_h + X_p = C_1 e^{-t} \begin{pmatrix} 3 \\ -1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -e^{2t} + \frac{19}{3} \\ -e^{2t} - \frac{5}{3} \end{pmatrix}.$$

Solution to problem 7. **a)** To write the equation as a system we introduce the variables $x = u$ and $y = u'$. This gives us $x' = u' = y$ and $y' = u'' = -cu' - u(1-u)(u-a) = -cy - x(1-x)(x-a)$, we hence get the system

$$\begin{cases} x' &= y \\ y' &= -cy - x(1-x)(x-a) \end{cases}.$$

b) The critical points are given by $x' = y' = 0$. To find the critical points we start by noticing that $x' = y$ immediately gives us $y = 0$. For the second equation we need $-cy - x(1-x)(x-a) = 0$, but $y = 0$ reduces this to $x(1-x)(x-a) = 0$ which has the solutions $x = 0$, $x = a$ and $x = 1$. Since $0 < a < 1$ the three solutions are distinct. The system hence has the three critical points

$$(0,0), \ (a,0) \text{ and } (1,0).$$

c) To determine the stability we start by computing the Jacobian of the system, we get

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ a - 2(1+a)x + 3x^2 & -c \end{pmatrix}.$$

We find the stability by studying the eigenvalues of the Jacobian for the three critical points. The characteristic polynomial is given by

$$\det(J(x,y) - \lambda I) = \lambda^2 + c\lambda - a + 2(1+a)x - 3x^2.$$

For the critical point $(0,0)$ we get

$$\det(J(0,0) - \lambda I) = \lambda^2 + c\lambda - a.$$

which has the roots

$$-\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + a}.$$

Since $\frac{c^2}{4} + a > 0$ the roots are real. Furthermore $\sqrt{\frac{c^2}{4} + a} > \sqrt{\frac{c^2}{4}} = \frac{c}{2}$ implies that one root is negative and the other positive, which means we have a saddle point, which is then unstable.

For the critical point $(a, 0)$ we get

$$\det(J(0,0) - \lambda I) = \lambda^2 + c\lambda + a - a^2$$

which has the roots

$$-\frac{c}{2} + \sqrt{\frac{c^2}{4} - a + a^2}.$$

By assumption $c^2 < 4a - 4a^2$, which is equivalent to $\frac{c^2}{4} - a + a^2 < 0$. We hence have complex eigenvalues, which gives us a spiral that is stable since $-\frac{c}{2} < 0$.

Finally for the critical point $(1, 0)$ we get

$$\det(J(0,0) - \lambda I) = \lambda^2 + c\lambda + a - 1$$

which has the roots

$$-\frac{c}{2} + \sqrt{\frac{c^2}{4} - a + 1}.$$

Since $a < 1$ we have $\frac{c^2}{4} - a + 1 > 0$ and the roots are real. As with the root critical point $(0, 0)$ we have $\sqrt{\frac{c^2}{4} - a + 1} > \sqrt{\frac{c^2}{4}} = \frac{c}{2}$, implying that one root is negative and the other positive. This means we have a saddle point, which is then unstable.

Solution to problem 8. **a)** The critical points are given by

$$\begin{cases} 0 &= x^4 + 8y^5 \\ 0 &= 2x^2 - 2y^3 \end{cases}.$$

We immediately see that $(0, 0)$ is a critical point. We want to determine if there are any other critical points. The second equation gives us $x^2 = y^3$. Inserted into the first equation this gives $y^6 = -8y^5$, which has the only non-zero solution $y = -8$. This gives $x^2 = (-8)^3$, but $(-8)^3 < 0$ so this has no real solutions. We conclude that the origin is the only critical point, since it is the only critical point it must also be isolated.

b) We make the ansatz $V(x, y) = ax^k + by^l$ for a Lyapunov function. To begin with we compute the derivative of V with respect to the system, we get

$$\frac{dV}{dt} = kax^{k-1}(x^4 + 8y^5) + lby^{l-1}(2x^2 - 2y^3) = kax^{k+3} + 8kax^{k-1}y^5 + 2lbx^2y^{l-1} - 2lby^{l+2}.$$

Since the goal is that $\frac{dV}{dt}$ should be either positive or negative definite we in particular want to remove the term $8kax^{k-1}y^5$ whose sign depend on y. We notice that with the right choice of parameters we can make it cancel with the $2lbx^2y^{l-1}$ term. More precisely, taking $k = 3$, $l = 6$ and $b = -2a$ we get

$$\frac{dV}{dt} = 3ax^6 + 24ax^2y^5 - 24ax^2y^6 + 24ay^8 = a(3x^6 + 24y^8).$$

We see with this choice of parameters the sign of $\frac{dV}{dt}$ only depends on the sign of a.

With the above choice of parameters we get

$$V(x, y) = a(x^3 - 2y^6)$$

which we note is neither positive nor negative definite.

From Liapunov's theorem we know that if $\frac{dV}{dt}$ is positive definite and V satisfies $V(0, 0) = 0$ and is positive at some point in every neighbourhood of the origin then the origin is unstable. We have that if $a > 0$ then $\frac{dV}{dt}$ is positive definite and we clearly have $V(0, 0) = 0$. It remains to show that V is positive at some point in every neighbourhood of the origin for in this case. Taking for example $a = 1$ gives

$$V(x, y) = x^3 - 2y^6$$

and this is clearly positive along the positive x-axis, which is included in every neighbourhood of the origin. We conclude that the above function satisfies the requirements and hence the origin is unstable.