English Version

1. (a) (3 p) Find the general solution, $y = y(x)$, of the ODE,

$$y' = \left(\frac{1}{x-a}\right) y + 1 - \frac{x-a}{x},$$

where a is a constant.

(b) (1 p) What is the solution in part (a) when $a = 1$ and $y(1/2) = 0$? Give the largest possible interval of the solution.

(c) (1 p) What is the solution in part (a) when $a = -3$ and $y(-4) = 0$? Give the largest possible interval of the solution.

2. (5 p) Find the general solution, $y = y(x)$, of the ODE,

$$y'' + 2y' + 3y = 2\cos(\sqrt{2}x) - e^{-x} + x^2, \quad -\infty < x < \infty.$$

3. (5 p) Find the general solution, $y = y(x)$, of the ODE,

$$xy'' + (2x-1)y' + (x-1)y = 0, \quad -\infty < x < \infty.$$

Hint: One solution is of the form $y(x) = e^{rx}$ for some constant r.

4. (a) (1 p) Consider the ODE,

$$\left(x^2 - \frac{1}{4}\right) y'' + 2xy' - 2y = 0, \quad y = y(x).$$

Prove that $x_0 = 0$ is an ordinary point of this ODE.

(b) (1 p) Argue that the ODE in part (a) has two linearly independent power series solutions, around $x_0 = 0$, which converge on $-\frac{1}{2} < x < \frac{1}{2}$.

(c) (3 p) Find two such linearly independent power series solutions. Give the full expressions for the power series, not just the first few terms. Form the general power series solution.

5. (5 p) Find the general solution, $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, of the system of ODEs,

$$\begin{array}{ll} x' & = 3x - y \\ y' & = 6x - 4y \end{array}, \quad -\infty < t < \infty.$$

Plot the phase portrait. Classify the critical point, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

6. (5 p) Find the critical points of the system of ODEs,

$$\begin{array}{rl} x' &= x - y^2 \\ y' &= -2x - 3y + xy \end{array} \quad , \quad \begin{array}{rl} x &= x(t) \\ y &= y(t) \end{array} \quad , \quad -\infty < t < \infty.$$

Classify each critical point.

7. (5 p) Prove that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an asymptotically stable critical point of the system of ODEs,

$$\begin{array}{rl} x' &= -2x^5 - 3y \\ y' &= 2x^3 - 3y^3 \end{array} \quad , \quad \begin{array}{rl} x &= x(t) \\ y &= y(t) \end{array} \quad , \quad -\infty < t < \infty.$$

8. (a) (2 p) Find the general solution, $y = y(x)$, of the following Euler equation on the interval $x > 0$:

$$x^2y'' + 7xy' + 9y = 0.$$

(b) (1 p) Find the general solution of the ODE in part (a) on the interval $x < 0$.

(c) (2 p) Is it possible to find a solution of the ODE in part (a) on the interval $-\infty < x < \infty$, other than the trivial solution where $y(x) = 0$ for all $-\infty < x < \infty$? Justify your answer.

GOOD LUCK!!