E1 - Exam solutions 2015/10/19

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Q1) (a) Solve the first order ODE

$$y' = \frac{1}{x-a} y + 1 - \frac{x-a}{x},$$

where a is a constant.

(b) What is the solution in part (a) when $a=1$ and $y(\frac{1}{2})=0$? Give the largest possible interval of the solution.

(c) What is the solution in part (a) when $a=3$ and $y(-4)=0$? Give the largest possible interval of the solution.

<u>Solution</u> (a)

$$y' = \frac{1}{x-a} y + 1 - \frac{x-a}{x}$$

$$\Rightarrow \; y' \underbrace{- \frac{1}{x-a}}_{P(x)} y = 1 - \frac{x-a}{x}$$

← a linear equation ⇒ integrating factor method

Integrating factor $= \exp\left(\int P(x)\,dx\right)$ ← ignore constants of integration and absolute values

(Note $\exp(x) = e^{x}$)

$$= \exp\left(\int\left(-\frac{1}{x-a}\right)dx\right)$$

$$= \exp\left(-\log(x-a)\right)$$

$$= \exp\left(\log((x-a)^{-1})\right) = (x-a)^{-1} = \frac{1}{x-a}$$

Recall $\quad y' - \frac{1}{x-a} y = 1 - \frac{x-a}{x}$

multiply both sides by $\frac{1}{x-a}$ $\Rightarrow$ $\quad \frac{1}{x-a} y' - \frac{1}{(x-a)^2} y = \frac{1}{x-a} - \frac{1}{x}$

product rule $\Rightarrow$ $\quad \frac{d}{dx}\left(\frac{1}{x-a} y \right) = \frac{1}{x-a} - \frac{1}{x}$

$\Rightarrow \quad \frac{1}{x-a} y = \int \left(\frac{1}{x-a} - \frac{1}{x} \right) dx + c$

$\Rightarrow \quad \frac{1}{x-a} y = \log|x-a| - \log|x| + c$

$\Rightarrow \quad y = (x-a)\log|x-a| - (x-a)\log|x| + c(x-a).$

<u>Check this works</u> :

$y = (x-a)\log|x-a| - (x-a)\log|x| + c(x-a)$ is a solution

$\Leftrightarrow \quad y'(x) = \frac{1}{x-a} y + 1 - \frac{x-a}{x}$

$\Leftrightarrow \quad (1)(\log|x-a|) + (x-a)\left(\frac{1}{x-a}\right) - (1)(\log|x|) - \frac{x-a}{x} + c(1)$

$= \frac{1}{x-a}\Big((x-a)\log|x-a| - (x-a)\log|x| + c(x-a) \Big) + 1 - \frac{x-a}{x}$

$\Leftrightarrow \quad \log|x-a| + 1 - \log|x| - \frac{x-a}{x} + c$

$= \quad \log|x-a| - \log|x| + c + 1 - \frac{x-a}{x}$ <u>OK</u>

(b) $a=1 \Rightarrow \quad y' = \frac{1}{x-1} y + 1 - \frac{x-1}{x}$

Also $y = (x-1)\log|x-1| - (x-1)\log|x| + c(x-1).$

$y(\frac{1}{2})=0 \Rightarrow (-\frac{1}{2})\log|-\frac{1}{2}| - (-\frac{1}{2})\log|-\frac{1}{2}| + c(-\frac{1}{2}) = 0$

$\Rightarrow -\frac{1}{2}\log(\frac{1}{2}) + \frac{1}{2}\log(\frac{1}{2}) - \frac{1}{2}c = 0$

$\Rightarrow c = 0$

Thus we have

$y' = \frac{1}{x-1} y + 1 - \frac{x-1}{x}$ ← RHS must be well defined in the interval of solution $\Rightarrow x \neq 0$ and $x \neq 1$

$y = (x-1)\log|x-1| - (x-1)\log|x|$, $y(\frac{1}{2}) = 1$

↑ interval of solution must contain $x = \frac{1}{2}$

The largest possible interval of the solution is $0 < x < 1$.

(c) $a=-3 \Rightarrow \quad y' = \frac{1}{x-(-3)} y + 1 - \frac{x-(-3)}{x}$

$\Rightarrow y' = \frac{1}{x+3} y + 1 - \frac{x+3}{x}.$

Also $y = (x-(-3))\log|x-(-3)| - (x-(-3))\log|x| + c(x-(-3))$

$\Rightarrow y = (x+3)\log|x+3| - (x+3)\log|x| + c(x+3)$

$y(-4)=0 \Rightarrow (-1)\log|-1| - (-1)\log|-4| + c(-1) = 0$

$\Rightarrow -\underbrace{\log(1)}_{=0} + \log(4) - c = 0$

$\Rightarrow c = \log(4)$

Thus we have

$$y' = \frac{1}{x+3}y + 1 - \frac{x+3}{x}$$ ← RHS must be well-defined in the interval of solution $\Rightarrow x \neq -3$ and $x \neq 0$

$$y = (x+3)\log|x+3| - (x+3)\log|x| + \log(4)(x+3), \quad y(-4) = 0$$

↑ interval of solution must contain $x = -4$

Thus the largest possible interval of solution is $-\infty < x < -3$.

Q2) Find the general solution of

$$y'' + 2y' + 3y = 2\cos(\sqrt{2}x) - e^{-x} + x^2, \quad -\infty < x < \infty$$

Solution: First find the general solution of

$$y'' + 2y' + 3y = 0.$$

$y = e^{rx}$ is a solution of this

$\Leftrightarrow y''(x) + 2y'(x) + 3y(x) = 0$ for all $-\infty < x < \infty$

$\Leftrightarrow (r^2e^{rx}) + 2(re^{rx}) + 3(e^{rx}) = 0$ for all $-\infty < x < \infty$

$\Leftrightarrow r^2 + 2r + 3 = 0$ ← characteristic equation

$\Leftrightarrow r = r_1$ or $r = r_2$, where

$$r_1 = \frac{-2 + \sqrt{(2)^2 - (4)(1)(3)}}{2} = -1 + \sqrt{2}i \quad \text{and} \quad r_2 = \frac{-2 - \sqrt{(2)^2 - (4)(1)(3)}}{2} = -1 - \sqrt{2}i$$

Thus $y_1(x) = e^{r_1 x} = e^{(-1+\sqrt{2}i)x}$ and $y_2(x) = e^{r_2 x} = e^{(-1-\sqrt{2}i)x}$

are two complex valued solutions of $y'' + 2y' + 3y = 0$.

Moreover they are linearly independent since they are not constant multiples of each other. The general complex solution is therefore

$c_1 y_1(x) + c_2 y_2(x)$, where c_1 and c_2 are <u>any</u> complex numbers. Note,

$$c_1 y_1(x) + c_2 y_2(x) = c_1 e^{(-1+\sqrt{2}i)x} + c_2 e^{(-1-\sqrt{2}i)x}$$

$$= c_1 e^{-x} e^{\sqrt{2}xi} + c_2 e^{-x} e^{-\sqrt{2}xi}$$

$$\overset{\text{Eulers}}{\underset{\text{formula}}{=}} c_1 e^{-x}\left(\cos(\sqrt{2}x) + i\sin(\sqrt{2}x)\right) + c_2 e^{-x}\left(\cos(\sqrt{2}x) - i\sin(\sqrt{2}x)\right)$$

$$= \underbrace{(c_1 + c_2)}_{c_3} \underbrace{e^{-x}\cos(\sqrt{2}x)}_{y_3(x)} + \underbrace{i(c_1 - c_2)}_{c_4} \underbrace{e^{-x}\sin(\sqrt{2}x)}_{y_4(x)}$$

Note that we can choose c_1 and c_2 such that c_3 and c_4 take <u>any</u> real values. Thus the general real solution of $y'' + 2y' + 3y = 0$ is

$c_3 y_3(x) + c_4 y_4(x)$, where c_3 and c_4 are <u>any</u> real numbers.

Next we find $\underline{a}$ solution, Y, of $y''+2y'+3y = \cancel{e^{x}\cos(\sqrt{2}x) - 2e^{-x}}$.

First guess $Y(x) = ae^{x}\cos(\sqrt{2}x) + be^{x}\sin(\sqrt{2}x) + ce^{-x}$,

where a, b, c constants.

$$\Rightarrow Y'(x) = a\left[(e^{x})(\cos(\sqrt{2}x)) + (e^{x})(-\sqrt{2}\sin(\sqrt{2}x))\right]$$
$$+ b\left[(e^{x})(\sin(\sqrt{2}x)) + (e^{x})(\sqrt{2}\cos(\sqrt{2}x))\right] + c(-e^{-x})$$

$$\Rightarrow Y''(x) = a\left[(e^{x})(\cos(\sqrt{2}x)) + 2(e^{x})(-\sqrt{2}\sin(\sqrt{2}x)) + (e^{x})(-2\cos(\sqrt{2}x))\right]$$
$$+ b\left[(e^{x})(\sin(\sqrt{2}x)) + 2(e^{x})(\sqrt{2}\cos(\sqrt{2}x)) + (e^{x})(-2\sin(\sqrt{2}x))\right] + c(e^{-x})$$

$$= a\left[-e^{x}\cos(\sqrt{2}x) - 2\sqrt{2}\,e^{x}\sin(\sqrt{2}x)\right]$$
$$+ b\left[-e^{x}\sin(\sqrt{2}x) + 2\sqrt{2}\,e^{x}\cos(\sqrt{2}x)\right] + ce^{-x}$$

$Y = Y(x)$ is a solution of $y''+2y'+3y = e^{x}\cos(\sqrt{2}x) - 2e^{-x}$

$\Leftrightarrow$ $Y''(x) + 2Y'(x) + 3Y(x) = e^{x}\cos(\sqrt{2}x) - 2e^{-x}$ for $\underline{all}$ $-\infty < x < \infty$

$\Leftrightarrow$ $a\left[e^{-x}\cos(\sqrt{2}x) - 2\sqrt{2}e^{x}\sin(\sqrt{2}x)\right] + b\left[e^{-x}\sin(\sqrt{2}x) + 2\sqrt{2}e^{x}\cos(\sqrt{2}x)\right] + ce^{-x}$

$+ 2a\left[e^{x}\cos(\sqrt{2}x) - \sqrt{2}e^{x}\sin(\sqrt{2}x)\right] + 2b\left[e^{x}\sin(\sqrt{2}x) + \sqrt{2}e^{x}\cos(\sqrt{2}x)\right] - 2ce^{-x}$

$+ 3ae^{x}\cos(\sqrt{2}x) + 3be^{x}\sin(\sqrt{2}x) + 3ce^{-x} = e^{x}\cos(\sqrt{2}x) - 2e^{-x}$

$\Leftrightarrow$ $e^{x}\cos(\sqrt{2}x)\left[a + 2\sqrt{2}b + 2a + 2\sqrt{2}b + 3a\right]$

$+ e^{x}\sin(\sqrt{2}x)\left[-2\sqrt{2}a + b - 2\sqrt{2}a + 2b + 3b\right] + 2ce^{-x}$

$= e^{x}\cos(\sqrt{2}x) - 2e^{-x}$ for $\underline{all}$ $-\infty < x < \infty$

Next we find a solution, Y, of

$$y'' + 2y' + 3y = 2\cos(\sqrt{2}x) - e^{-x} + x^2, \quad -\infty < x < \infty.$$

Guess $Y(x) = a\cos(\sqrt{2}x) + b\sin(\sqrt{2}x) + ce^{-x} + dx^2 + ex + f$

$$\Rightarrow Y'(x) = a(-\sqrt{2}\sin(\sqrt{2}x)) + b(\sqrt{2}\cos(\sqrt{2}x)) + c(-e^{-x}) + d(2x) + e(1) + 0$$

$$= -\sqrt{2}a\sin(\sqrt{2}x) + \sqrt{2}b\cos(\sqrt{2}x) - ce^{-x} + 2dx + e$$

$$\Rightarrow Y''(x) = -\sqrt{2}a(\sqrt{2}\cos(\sqrt{2}x)) + \sqrt{2}b(-\sqrt{2}\sin(\sqrt{2}x)) - c(-e^{-x}) + 2d(1) + 0$$

$$= -2a\cos(\sqrt{2}x) - 2b\sin(\sqrt{2}x) + ce^{-x} + 2d.$$

$Y = Y(x)$ is a solution

$$\Leftrightarrow Y''(x) + 2Y'(x) + 3Y = 2\cos(\sqrt{2}x) - e^{-x} + x^2 \quad \text{for } \underline{\text{all}} \ -\infty < x < \infty$$

$$\Leftrightarrow -2a\cos(\sqrt{2}x) - 2b\sin(\sqrt{2}x) + ce^{-x} + 2d$$
$$+ 2(-\sqrt{2}a\sin(\sqrt{2}x) + \sqrt{2}b\cos(\sqrt{2}x) - ce^{-x} + 2dx + e)$$
$$+ 3(a\cos(\sqrt{2}x) + b\sin(\sqrt{2}x) + ce^{-x} + dx^2 + ex + f)$$
$$= 2\cos(\sqrt{2}x) - e^{-x} + x^2 + 1 \quad \text{for } \underline{\text{all}} \ -\infty < x < \infty$$

$$\Leftrightarrow (-2a + 2\sqrt{2}b + 3a)\cos(\sqrt{2}x) + (-2b - 2\sqrt{2}a + 3b)\sin(\sqrt{2}x)$$
$$+ 2ce^{-x} + 3dx^2 + (4d + 3e)x + (2d + 2e + 3f)$$
$$= 2\cos(\sqrt{2}x) - e^{-x} + x^2 \quad \text{for } \underline{\text{all}} \ -\infty < x < \infty$$

$$\iff \left.\begin{array}{l} -2a + 2\sqrt{2}b + 3a = 2 \\ -2b - 2\sqrt{2}a + 3b = 0 \end{array}\right\} \Rightarrow \begin{array}{l} a + 2\sqrt{2}b = 2 \\ -2\sqrt{2}a + b = 0 \end{array} \Rightarrow \begin{array}{l} a + 2\sqrt{2}(2\sqrt{2}a) = 2 \\ b = 2\sqrt{2}a \end{array} \quad \begin{array}{l} 9a = 2 \\ \Rightarrow b = 2\sqrt{2}a \\ \Rightarrow a = 2/9 \\ \quad b = 4\sqrt{2}/9 \end{array}$$

$$2c = -1 \Rightarrow c = -1/2$$

$$\left.\begin{array}{l} 3d = 1 \\ 4d + 3e = 0 \\ 2d + 2e + 3f = 0 \end{array}\right\} \Rightarrow \begin{array}{l} d = 1/3 \\ 4(1/3) + 3e = 0 \\ 2(1/3) + 2e + 3f = 0 \end{array} \Rightarrow \begin{array}{l} d = 1/3 \\ e = -4/9 \\ 2(1/3) + 2(-4/9) + 3f = 0 \end{array} \Rightarrow \begin{array}{l} d = 1/3 \\ e = -4/9 \\ f = \frac{2}{27} \end{array}$$

$$\Rightarrow Y(x) = a\cos(\sqrt{2}x) + b\sin(\sqrt{2}x) + ce^{-x} + dx^2 + ex + 1$$

$$= \frac{2}{9}\cos(\sqrt{2}x) + \frac{4\sqrt{2}}{9}\sin(\sqrt{2}x) - \frac{1}{2}e^{-x} + \frac{1}{3}x^2 - \frac{4}{9}x + \frac{2}{27}.$$

The general solution of $y'' + 2y' + 3y = 2\cos(\sqrt{2}x) - e^{-x} + x^2$ is

$$\underbrace{c_3 y_3 + c_4 y_4}_{\text{general solution of } y''+2y'+3y=0} + \underbrace{Y}_{\text{a solution of } y''+2y'+3y = 2\cos(\sqrt{2}x) - e^{-x} + x^2}.$$

~~Q3) Find the general solution of~~

~~$(x^2+2x-1)y'' + (x^2-3)y' - (2x+2)y = 0.$~~

~~Hint: One solution is of the form " $y(x) = e^{rx}$ " for some constant r.~~

~~Solution) $y(x) = ax + b$ is a solution~~

~~$\iff (x-1)y_1''(x) + (x-2)y_1'(x) - y(x) = 0$ for all $-\infty < x < \infty$~~

Q3) Find the general solution of

$$xy'' + (2x-1)y' + (x-1)y = 0, \qquad -\infty < x < \infty.$$

Hint : One solution is of the form $y(x) = e^{rx}$ for some constant r.

Solution : $y(x) = e^{rx}$ is a solution

$\Leftrightarrow x\,y''(x) + (2x-1)y'(x) + (x-1)y(x) = 0$ for all $-\infty < x < \infty$

$\Leftrightarrow x(r^2e^{rx}) + (2x-1)(re^{rx}) + (x-1)(e^{rx}) = 0$ for all $-\infty < x < \infty$

$\Leftrightarrow x(r^2) + (2x-1)(r) + (x-1) = 0$ for all $-\infty < x < \infty$

$\Leftrightarrow x(r^2+2r+1) + (-r-1) = 0$ for all $-\infty < x < \infty$

$\Leftrightarrow r^2+2r+1 = 0$ and $-r-1 = 0$

$\Leftrightarrow (r+1)(r+1) = 0$ and $r+1 = 0$

$\Leftrightarrow r = -1.$

Thus $y_1(x) = e^{-x}$ is one solution.

reduction of order

Guess a second solution of the form,

$$y_2 = y_1 v, \quad \text{where } v = v(x).$$

$y_2 = y_1 v$ is a solution

$\Leftrightarrow x y_2'' + (2x-1)y_2' + (x-1)y_2 = 0$

$\Leftrightarrow x(y_1''v + 2y_1'v' + y_1v'') + (2x-1)(y_1'v + y_1v') + (x-1)(y_1v) = 0$

$\Leftrightarrow \underbrace{(xy_1'' + (2x-1)y_1' + (x-1)y_1)}_{=0}v + (2xy_1' + (2x-1)y_1)v' + xy_1v'' = 0$

$\Leftrightarrow (2xy_1' + (2x-1)y_1)\, v' + xy_1 v'' = 0$

$y_1(x) = e^{-x}$

$\Leftrightarrow \quad (2x(-e^{-x}) + (2x-1)(e^{-x}))\, v' + x(e^{-x})\, v'' = 0$

$\Leftrightarrow \quad (2x(-1) + (2x-1))\, v' + xv'' = 0$

$\Leftrightarrow \quad xv'' - v' = 0.$

Note that $v(x) = x^2$ is a solution of the above equation. Thus $y_2(x) = x^2 y_1(x) = x^2 e^{-x}$ is a second solution of $xy'' + (2x-1)y' + (x-1)y = 0$.

Moreover, y_1 and y_2 are linearly independent since they are not constant multiples of each other. The general solution is therefore

$y = c_1 y_1 + c_2 y_2$, where c_1 and c_2 are any real values.

Q4) Consider the ODE

$$(x^2 - \tfrac{1}{4})y'' + 2xy' - 2y = 0.$$

Prove that $x_0 = 0$ is an ordinary point. Argue that the above ODE has two linearly independent power series solutions which converge on $-\frac{1}{2} < x < \frac{1}{2}$. Find two such independent power series solutions and form the general power series solution.

Solution: Write as

$$y'' + \underbrace{\left(\frac{2x}{x^2 - \frac{1}{4}}\right)}_{P(x)} y' + \underbrace{\left(\frac{-2}{x^2 - \frac{1}{4}}\right)}_{q(x)} y = 0$$

$$\Rightarrow P(x) = \frac{2x}{x^2 - \frac{1}{4}} = 2x\left(\frac{1}{x^2 - \frac{1}{4}}\right) = (2x)(-4)\left(\frac{1}{1 - 4x^2}\right)$$

$\frac{1}{1-x}$ has a power series expansion around $x_0 = 0$ which converges on $-1 < x < 1 \Rightarrow \frac{1}{1-4x^2}$ has a power series expansion around $x_0 = 0$ which converges on $-1 < 4x^2 < 1 \Rightarrow -\frac{1}{4} < x^2 < \frac{1}{4} \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$.

$\Rightarrow P(x)$ has a power series expansion which converges on $-\frac{1}{2} < x < \frac{1}{2}$. Similarly for $q(x)$.

Thus $x_0 = 0$ is an ordinary point. Thus the ODE has two linearly independent power series solutions which converge on the same interval, $-\frac{1}{2} < x < \frac{1}{2}$

$$y(x) = \sum_{j=0}^{\infty} a_j x^j \quad \text{is a solution on} \quad -\tfrac{1}{2} < x < \tfrac{1}{2}$$

$$\Leftrightarrow (x^2 - \tfrac{1}{4}) y''(x) + 2x\, y'(x) - 2y(x) = 0 \quad \text{for } \underline{\underline{\text{all}}} \quad -\tfrac{1}{2} < x < \tfrac{1}{2}$$

$$\Leftrightarrow (x^2 - \tfrac{1}{4}) \left[\sum_{j=0}^{\infty} a_j\, j(j-1)\, x^{j-2} \right] + 2x \left[\sum_{j=0}^{\infty} a_j (j) x^{j-1} \right]$$

$$- 2 \left[\sum_{j=0}^{\infty} a_j x^j \right] = 0 \quad \text{for } \underline{\underline{\text{all}}} \quad -\tfrac{1}{2} < x < \tfrac{1}{2}$$

$$\Leftrightarrow \left[\sum_{j=0}^{\infty} a_j (j)(j-1) x^j - \underbrace{\tfrac{1}{4} \sum_{j=0}^{\infty} a_j (j)(j-1) x^{j-2}}_{j=0,1 \text{ terms equal } 0.} \right] + 2 \left[\sum_{j=0}^{\infty} a_j (j) x^j \right]$$

$$- 2 \left[\sum_{j=0}^{\infty} a_j x^j \right] = 0 \quad \text{for } \underline{\underline{\text{all}}} \quad -\tfrac{1}{2} < x < \tfrac{1}{2}$$

$$\Leftrightarrow \left[\sum_{j=0}^{\infty} a_j (j)(j-1) x^j - \tfrac{1}{4} \sum_{j=0}^{\infty} a_{j+2} (j+2)(j+1) x^j \right] + 2 \left[\sum_{j=0}^{\infty} a_j (j) x^j \right]$$

$$- 2 \left[\sum_{j=0}^{\infty} a_j x^j \right] = 0 \quad \text{for } \underline{\underline{\text{all}}} \quad -\tfrac{1}{2} < x < \tfrac{1}{2}$$

$$\Leftrightarrow \sum_{j=0}^{\infty} b_j x^j = 0 \quad \text{for } \underline{\underline{\text{all}}} \quad -\tfrac{1}{2} < x < \tfrac{1}{2}. \quad \ldots \circledast$$

where $b_j = a_j (j)(j-1) - \tfrac{1}{4} a_{j+2} (j+2)(j+1) + 2a_j (j) - 2a_j$

for all $j = 0, 1, 2 \ldots$

$$\Rightarrow b_j = a_j \left(j(j-1) + 2j - 2 \right) - \tfrac{1}{4} a_{j+2} (j+2)(j+1)$$

$$\Rightarrow b_j = a_j (j^2 + j - 2) - \tfrac{1}{4} a_{j+2} (j+2)(j+1)$$

$$\Rightarrow b_j = a_j (j+2)(j-1) - \tfrac{1}{4} a_{j+2} (j+2)(j+1) \quad \text{for} \quad j = 0, 1, 2 \ldots$$

Note, $\circledast$ is true whenever $\boxed{b_j = 0}$ for $\underline{\underline{\text{all}}}$ $j = 0, 1, 2 \ldots$

$b_j = 0$

$\Leftrightarrow a_j(j+2)(j-1) - \frac{1}{4} a_{j+2}(j+2)(j+1) = 0 \leftarrow$ recurrence relation

$\Leftrightarrow a_{j+2} = 4 \frac{j-1}{j+1} a_j \quad$ for $\quad j = 0, 1, 2 \ldots$

j	$a_{j+2} = 4 \frac{j-1}{j+1} a_j$
0	$a_2 = 4\left(\frac{-1}{1}\right) a_0 = -4 a_0$
1	$a_3 = 4\left(\frac{0}{2}\right) a_1 = 0$
2	$a_4 = 4\left(\frac{1}{3}\right) a_2 = (4)^2 \left(\frac{1}{3}\right)\left(\frac{-1}{1}\right) a_0 = -(4)^2\left(\frac{1}{3}\right) a_0$
3	$a_5 = (4)\left(\frac{2}{4}\right) a_3 = 0$
4	$a_6 = 4\left(\frac{3}{5}\right) a_4 = (4)^3 \left(\frac{3}{5}\right)\left(\frac{1}{3}\right)\left(\frac{-1}{1}\right) a_0 = -(4)^3\left(\frac{1}{5}\right) a_0$
5	$a_7 = 4\left(\frac{4}{6}\right) a_5 = 0$
6	$a_8 = 4\left(\frac{5}{7}\right) a_6 = (4)^4 \left(\frac{5}{7}\right)\left(\frac{3}{5}\right)\left(\frac{1}{3}\right)\left(\frac{-1}{1}\right) a_0 = -(4)^4\left(\frac{1}{7}\right) a_0$
$\vdots$	$\vdots$

no restrictions on a_0 and a_1

Thus $a_j = 0$ for $j = 3, 5, 7 \ldots$

$$a_j = -(4)^{j/2} \frac{1}{j-1} a_0 \quad \text{for} \quad j = 2, 4, 6 \ldots$$

Thus $y(x) = \sum_{j=0}^{\infty} a_j x^j = a_0 + a_1 x + \sum_{j=2,4,6\ldots} a_j x^j + \underbrace{\sum_{j=3,5,7\ldots} a_j x^j}_{=0}$

$$\Rightarrow y(x) = a_0 + a_1 x + \sum_{j=2,4,6\ldots} -(4)^{j/2} \frac{1}{j-1} a_0 x^{2j}$$

$$= a_0 + a_1 x + \sum_{j=1}^{\infty} -(4)^j \frac{1}{2j-1} a_0 x^{2j}$$

$$\Rightarrow y(x) = a_0 \underbrace{\left[1 - \sum_{j=1}^{\infty} \frac{(4)^j x^{2j}}{2j-1}\right]}_{y_1} + a_1 \underbrace{[x]}_{y_2}$$

Note that there are no restrictions on a_0 and a_1. Thus y_1 and y_2 are two power series solutions. Moreover y_1 and y_2 are linearly independent since they are not constant multiples of each other. Finally, since a_0 and a_1 are <u>any</u> real constants, and since y_1 and y_2 are linearly independent, $y = a_0 y_1 + a_1 y_2$ is the general power series solution.

Q5) Find the general solution of

$$x' = 3x - y, \quad y' = 6x - 4y, \qquad x = x(t), \quad y = y(t), \qquad -\infty < t < \infty.$$

Plot the phase portrait. Classify the critical point $\binom{0}{0}$.

Solution: $X' = \begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix} X$, where $X = \binom{x}{y}$.

$X(t) = \binom{v_1}{v_2} e^{\lambda t}$ is a solution with $\binom{v_1}{v_2} \neq \binom{0}{0}$

$\Leftrightarrow$ λ is an eigenvalue of $\begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix}$ with e-vector $\binom{v_1}{v_2} \neq \binom{0}{0}$

$\Leftrightarrow$ $\underbrace{\begin{vmatrix} 3-\lambda & -1 \\ 6 & -4-\lambda \end{vmatrix}}_{\text{characteristic polynomial}} = 0$ and $\begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix}\binom{v_1}{v_2} = \lambda \binom{v_1}{v_2}$

$\Leftrightarrow$ $(3-\lambda)(-4-\lambda) - (6)(-1) = 0$ and $\begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix}\binom{v_1}{v_2} = \lambda \binom{v_1}{v_2}$

$\Leftrightarrow$ $-12 - 3\lambda + 4\lambda + \lambda^2 + 6 = 0$ and $\begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix}\binom{v_1}{v_2} = \lambda \binom{v_1}{v_2}$

$\Leftrightarrow$ $\lambda^2 + \lambda - 6 = 0$ and $\begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix}\binom{v_1}{v_2} = \lambda \binom{v_1}{v_2}$

$\Leftrightarrow$ $(\lambda+3)(\lambda-2) = 0$ and $\begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix}\binom{v_1}{v_2} = \lambda \binom{v_1}{v_2}$

$\Leftrightarrow$ $\lambda = \lambda_1$ and $\begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix}\binom{v_1}{v_2} = \lambda_1 \binom{v_1}{v_2}$, where $\lambda_1 = 2$,

OR

$\lambda = \lambda_2$ and $\begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix}\binom{v_1}{v_2} = \lambda_2 \binom{v_1}{v_2}$, where $\lambda_2 = -3$.

$\lambda_1 = 2$: $\begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\Leftrightarrow \begin{array}{l} 3v_1 - v_2 = 2v_1 \\ 6v_1 - 4v_2 = 2v_2 \end{array} \Leftrightarrow v_1 = v_2.$$

Take $v_2 = 1 \Rightarrow v_1 = 1$. Thus $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix}$ with eigenvalue $\lambda_1 = 2$.

Thus $X_1(t) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{\lambda_1 t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$ is <u>a</u> solution of $X' = \begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix} X$.

$\lambda_2 = -3$: $\begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -3\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Leftrightarrow \begin{array}{l} 3v_1 - v_2 = -3v_1 \\ 6v_1 - 4v_2 = -3v_2 \end{array} \Leftrightarrow 6v_1 = v_2$

Take $v_1 = 1 \Rightarrow v_2 = 6$. Thus $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix}$ with e-value $\lambda_1 = -3$. Thus $X_2(t) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{\lambda_2 t} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} e^{-3t}$ is <u>a</u> solution of $X' = \begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix} X$.

Note that X_1 and X_2 are linearly independent since the are not constant multiples of each other.

The general solution of $X' = \begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix} X$ is therefore

$$X(t) = c_1 X_1(t) + c_2 X_2(t) \quad \leftarrow \text{where } c_1 \text{ and } c_2 \text{ are any real constants}$$

$$= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_1 \begin{pmatrix} 1 \\ 6 \end{pmatrix} e^{-3t}$$

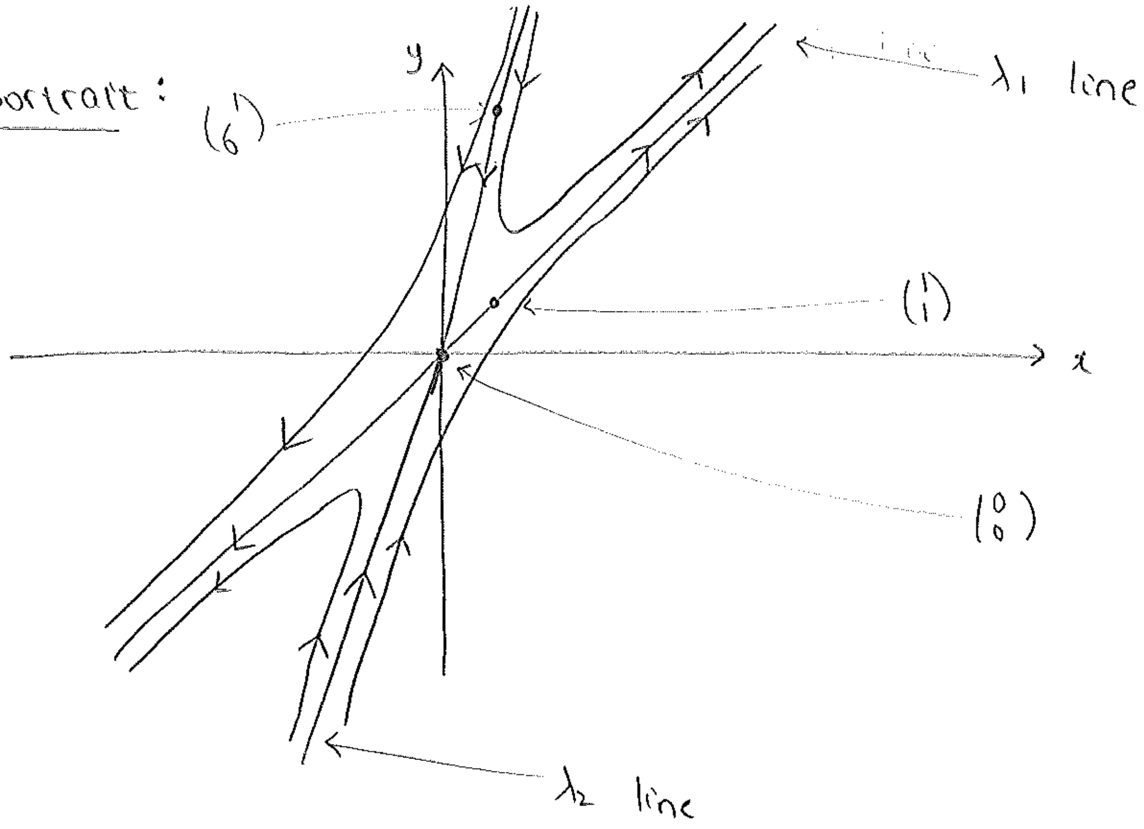
$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$\lambda_1 = 2$

$\lambda_2 = -3.$

Note $\underline{\lambda_1 > 0}$ and $\underline{\lambda_2 < 0} \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a saddle point.

Phase portrait:



Q6) Consider the system, $\begin{aligned} x' &= x - y^2 \\ y' &= -2x - 3y + xy \end{aligned}$, $\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$, $-\infty < t < \infty$.

Argue that the system is locally linear in a neighbourhood of each of its critical points. Find and classify the critical points.

Solution:

$$x' = x - y^2 \leftarrow F(x,y)$$
$$y' = -2x - 3y + xy \leftarrow G(x,y)$$

The system is locally linear in the neighbourhood of each critical point since F and G have continuous partial derivatives up to order 2.

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is a critical point

$$\Leftrightarrow \begin{cases} F(x_0, y_0) = 0 \\ G(x_0, y_0) = 0 \end{cases} \Leftrightarrow \begin{cases} x_0 - y_0^2 = 0 \\ -2x_0 - 3y_0 + x_0 y_0 = 0 \end{cases} \Leftrightarrow \begin{cases} x_0 = y_0^2 \\ -2x_0 - 3y_0 + x_0 y_0 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_0 = y_0^2 \\ -2(y_0^2) - 3y_0 + (y_0^2)y_0 = 0 \end{cases} \Leftrightarrow \begin{cases} x_0 = y_0^2 \\ y_0(-2y_0 - 3 + y_0^2) = 0 \end{cases} \Leftrightarrow \begin{cases} x_0 = y_0^2 \\ y_0(y_0^2 - 2y_0 - 3) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_0 = y_0^2 \\ y_0(y_0 - 3)(y_0 + 1) = 0 \end{cases} \Leftrightarrow \begin{cases} x_0 = y_0^2 \\ y_0 = 0 \end{cases} \text{ OR } \begin{cases} x_0 = y_0^2 \\ y_0 = 3 \end{cases} \text{ OR } \begin{cases} x_0 = y_0^2 \\ y_0 = -1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_0 = 0 \\ y_0 = 0 \end{cases} \text{ OR } \begin{cases} x_0 = 9 \\ y_0 = 3 \end{cases} \text{ OR } \begin{cases} x_0 = 1 \\ y_0 = -1 \end{cases}$$

$\underline{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$:

$$J(0,0) = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix} \Bigg|_{(0,0)}$$

$$= \begin{pmatrix} 1 & -2y \\ -2+y & -3+x \end{pmatrix} \Bigg|_{(0,0)}$$

$$\Rightarrow J(0,0) = \begin{pmatrix} 1 & 0 \\ -2 & -3 \end{pmatrix}$$

$F(x,y) = x - y^2$

$\Rightarrow \frac{\partial F}{\partial x} = 1$

$\frac{\partial F}{\partial y} = -2y$

$G(x,y) = -2x - 3y + xy$

$\frac{\partial G}{\partial x} = -2 + y$

$\frac{\partial G}{\partial y} = -3 + x$

$J(0,0)$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -3$ $\Rightarrow \lambda_1 > 0$ and $\lambda_2 < 0$

$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a saddle point (unstable).

$\underline{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}}$:

$$J(9,3) = \begin{pmatrix} 1 & -2y \\ -2+y & -3+x \end{pmatrix} \Bigg|_{(9,3)} = \begin{pmatrix} 1 & -6 \\ 1 & 6 \end{pmatrix}$$

λ is an eigenvalue of $J(9,3) = \begin{pmatrix} 1 & -6 \\ 1 & 6 \end{pmatrix}$

$$\Leftrightarrow \begin{vmatrix} 1-\lambda & -6 \\ 1 & 6-\lambda \end{vmatrix} = 0 \Leftrightarrow (1-\lambda)(6-\lambda) - (1)(-6) = 0$$

$\Leftrightarrow \lambda^2 - 7\lambda + 12 = 0 \quad \Leftrightarrow \quad (\lambda-4)(\lambda-3) = 0$

$\Leftrightarrow \lambda = \lambda_3$ or $\lambda = \lambda_4$ where $\lambda_3 = 4$ and $\lambda_4 = 3$.

Thus $\lambda_3 > 0$ and $\lambda_4 > 0$ and $\lambda_3 \neq \lambda_4$

$\Rightarrow \begin{pmatrix} 9 \\ 3 \end{pmatrix}$ is an unstable node.

$\underline{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$: $J(1,-1) = \begin{pmatrix} 1 & -2y \\ -2+y & -3+x \end{pmatrix} \Bigg|_{(1,-1)}$

$= \begin{pmatrix} 1 & 2 \\ -3 & -2 \end{pmatrix}$.

λ is an eigenvalue of $J(1,-1) = \begin{pmatrix} 1 & 2 \\ -3 & -2 \end{pmatrix}$

$\Leftrightarrow \begin{vmatrix} 1-\lambda & 2 \\ -3 & -2-\lambda \end{vmatrix} = 0 \quad \Leftrightarrow \quad (1-\lambda)(-2-\lambda) - (-3)(2) = 0$

$\Leftrightarrow -2 - \lambda + 2\lambda + \lambda^2 + 6 = 0 \quad \Leftrightarrow \quad \lambda^2 + \lambda + 4 = 0$

$\Leftrightarrow \lambda = \lambda_5$ or $\lambda = \lambda_6$, where

$\lambda_5 = \dfrac{-1 + \sqrt{(1)^2 - (4)(1)(4)}}{2}$ and $\lambda_6 = \dfrac{-1 - \sqrt{(1)^2 - (4)(1)(4)}}{2}$

$= \dfrac{-1 + \sqrt{-15}}{2}$ $\qquad = -\frac{1}{2} - i\,\frac{1}{2}\sqrt{15}$

$= -\frac{1}{2} + i\,\frac{1}{2}\sqrt{15}$

Thus $\lambda_5 = \alpha + i\beta$ and $\lambda_6 = \alpha - i\beta$ with $\alpha < 0$

$\Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an asymptotically stable spiral point.

here one also needs to explain that the portrait and stability types for the original non-linear system is the same as for the linearized system

Q7) Prove that $(0,0)$ is an asymptotically critical point of

$$x' = -2x^5 - 3y \leftarrow F(x,y) \qquad x = x(t)$$
$$y' = 2x^3 - 3y^3 \leftarrow G(x,y), \qquad y = y(t)$$

<u>Solution</u>: $\binom{0}{0}$ is a critical point since

$$F(0,0) = 0.$$
$$G(0,0) = 0$$

We use Liapounovs Second method to examine stability. Guess a Liapounov function of the form

$$V(x,y) = ax^{2k} + by^{2l}$$ where a,b: constants, k,l integers.

Note that $\dot{V}(x,y) = \left(\frac{\partial V}{\partial x}\right)F + \left(\frac{\partial V}{\partial y}\right)G$

$$= (2kax^{2k-1})(-2x^5 - 3y) + (2lby^{2l-1})(2x^3 - 3y^3)$$

$$\Rightarrow \dot{V}(x,y) = -4ka\underbrace{x^{2k+4}}_{\text{even power} \Rightarrow x^{2k+4} \geq 0} \underbrace{-6kax^{2k-1}y + 4lbx^3y^{2l-1}}_{\text{Choose } a,b,k,l \text{ such that this equals } 0.} - 6lb\underbrace{y^{2l+2}}_{\text{even power} \Rightarrow x^{2l+2} \geq 0}$$

First choose k, l such that $x^{2k-1}y = x^3y^{2l-1}$

$\Rightarrow k = 2$ and $l = 1$.

$\Rightarrow 6kax^{2k-1}y + 4lbx^3y^{2l-1}$

$= -12ax^3y + 4bx^3y.$

Then choose $a = 1$ and $b = 3$ so that

$-12ax^3y + 4bx^3y$

$= -12x^3y + 12x^3y = 0$

Thus, taking $a=1, b=3, K=2$ and $l=1$ gives

$$V(x,y) = ax^{2k} + by^{2l}$$
$$= x^4 + 3y^2 \leftarrow$$ positive definite in all neighbourhoods of $\binom{0}{0}$

and

$$\dot{V}(x,y) = -4kax^{2k+4} - 6lby^{2l+2}$$
$$= -8x^8 - 18y^4 \leftarrow$$ negative definite in all neighbourhoods of $\binom{0}{0}$.

Liapounovs second method thus implies that $\binom{0}{0}$ is an asymptotically stable critical point

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Q8) Consider the Euler equation

$$x^2 y'' + 7xy' + 9y = 0.$$

(a) Find the general solution of the equation on the interval $x>0$. (b) Find the general solution on the interval $x<0$. (c) Is it possible to find a solution on the interval $-\infty < x < \infty$, other than the trivial solution $y=0$? Justify your answer.

Solution: $y(x) = x^r$ is a solution for $x>0$

$$\Leftrightarrow x^2 y''(x) + 7x y'(x) + 9y(x) = 0 \quad \text{for } \underline{\text{all}} \ x>0$$
$$\Leftrightarrow x^2(r(r-1)x^{r-2}) + 7x(rx^{r-1}) + 9(x^r) = 0$$

for $\underline{\text{all}}$ $x>0$

indicial equation

$\Leftrightarrow \quad r(r-1) + 7r + 9 = 0 \quad \Leftrightarrow \quad r^2 + 6r + 9 = 0$

$\Leftrightarrow \quad (r+3)(r+3) = 0 \quad \Leftrightarrow \quad r = r_1$ or $r = r_2$, where $r_1 = -3$ and $r_2 = -3$.

The indicial equation has a real-valued repeated root, $r_1 = r_2 = -3$. Thus $y_1(x) = x^{r_1} = x^{-3}$ is one solution, and $y_2(x) = x^{r_1}\log(x) = x^{-3}\log(x)$ is a second linearly independent solution.

The general solution of $x^2y'' + 7xy' + 9y = 0$ for $x > 0$ is $y = C_1 y_1 + C_2 y_2$, where C_1 and C_2 are any real values.

(b) Guess from part (a) that the general solution of $x^2y'' + 7xy' + 9y = 0$ for $x < 0$ is $y = C_3 y_3 + C_4 y_4$, where $y_3(x) = x^{-3}$ and $y_4(x) = x^{-3}\log|x|$, and C_3 and C_4 take any real values. This is true if and only if y_3 and y_4 are linearly independent solutions for $x < 0$.

Check: $y_3(x) = x^{-3}$ is a solution for $x < 0$

$\Leftrightarrow \quad x^2 y_3''(x) + 7x y_3'(x) + 9y_3(x) = 0$ for all $x < 0$

$\Leftrightarrow \quad x^2(12x^{-5}) + 7x(-3x^{-4}) + 9(x^{-3}) = 0$ for all $x < 0$

$\Leftrightarrow \quad x^{-3}(12 - 21 + 9) = 0$ for all $x < 0$ OK

Also $y_4(x) = x^{-3}\log|x| \Rightarrow y_4'(x) = -3x^{-4}\log|x| + x^{-4}$

$\Rightarrow y_4''(x) = 12x^{-5}\log|x| - 3x^{-5} - 4x^{-5}$

$y_4(x) = x^{-3}\log|x|$ is a solution for $x<0$

$\Leftrightarrow$ $x^2 y_4''(x) + 7x y_4'(x) + 9y_4(x) = 0$ for <u>all</u> $x<0$

$\Leftrightarrow$ $x^2(12x^{-5}\log|x| - 7x^{-5}) + 7x(-3x^{-4}\log|x| + x^{-4}) + 9(x^{-3}\log|x|)$

for <u>all</u> $x<0$

$\Leftrightarrow$ $(12x^{-3}\log|x| - 7x^{-3}) + (-21x^{-3}\log|x| + 7x^{-3}) + 9(x^{-3}\log|x|)$

for <u>all</u> $x<0$

$\Leftrightarrow$ $(12-21+9)x^{-3}\log|x| + (-7+7)x^{-3} = 0$ for all $x<0$

<u>OK</u>

(c) <u>NO</u>.

We justify this by arguing by contradiction.

Assume that $y \neq 0$ is a solution of

$x^2 y'' + 7xy' + 9y = 0$ on $-\infty < x < \infty$.

It follows from part (a) that there exists <u>some</u> constants, C_1 and C_2, for which $y(x) = C_1 y_1(x) + C_2 y_2(x)$ for <u>all</u> $x>0$. Also, it follows from part (b) that there exists constants, C_3 and C_4, for which $y(x) = C_3 y_3(x) + C_4 y_4(x)$ for <u>all</u> $x<0$.

$$\Rightarrow y(x) = \begin{cases} C_1 y_1(x) + C_2 y_2(x) & \text{for } x>0 \\ C_3 y_3(x) + C_4 y_4(x) & \text{for } x<0. \end{cases}$$

Next note that $y_1(x) = x^{-3}$ is not well-defined at $x=0$.

Similarly $y_2(x)$, $y_3(x)$, $y_4(x)$ are not well-defined at $x=0$

Finally note that, since $y \neq 0$, that at least one of c_1, c_2, c_3, c_4 must be non-zero. Thus y is not well-defined at $x=0$. Thus we have a contradiction, and so our assumption that $x^2y'' + 7xy' + 9y = 0$ has a solution on $-\infty < x < \infty$ is false.