**UPPSALA UNIVERSITET**
**Matematiska institutionen**
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Prov i matematik
Ordinära differentialekvationer I
1MA032, Q2, 2015-12-14

*Time: 14.00 – 19.00. Tools allowed: only materials for writing.*
*Please provide full explanations and calculations in order to get full credit.*
*The exam consists of* **8 problems** *of* **10 points** *each for a total of* **80 points**. *For grades 3,4, and 5, one should obtain 36, 50, and 64 points, respectively.*
*Good luck and have fun!*

**1.** (a) (2 points) Complete the following definition: differential equation

$$P(x, y) + Q(x, y)y' = 0$$

is called exact if there exists a function $\psi(x, y)$ such that...

(b) (8 points) Find the general solution of the ODE

$$(xe^{xy} + 2015y)y' = 2016x - ye^{xy}.$$

**2.** Parts (a)–(d) are unrelated.

(a) (3 points) Find the general solution of the ODE $y'(t) = 1/t$ on the domain $t < 0$.

(b) (2 points) Complete the definition: a collection of functions $\phi_1(t), \dots, \phi_n(t)$ is called linearly dependent on an interval $\alpha < t < \beta$ if...

(c) (2 points) Rewrite the integral equation $y(t) - \int_2^t (1 + s + e^{y(s)^2})ds = 0$ as an ODE together with an initial condition.

(d) (3 points) Using the Sturm separation theorem, prove that zeros of functions $\sin x + 14 \cos x$ and $12 \sin x + 2015 \cos x$ are distinct and occur alternately (no credit if Sturm theorem is not used).

ignore part (d)
as it wasn't part
of our course

**3.** (a) (5 points) Solve the initial value problem

$$y''(x) - 2y'(x) + y(x) = 0, \qquad -\infty < x < \infty,$$
$$y(0) = 2015, \quad y'(0) = 2016.$$

(b) (5 points) Find the general solution of the ODE

$$y''(x) - 2y'(x) + y(x) = e^x, \qquad -\infty < x < \infty.$$

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**4.** Consider the ODE
$$xy'' + y' - y = 0.$$

(a) (2 points) Classify (ordinary/regular singular/irregular singular) the point $x = 0$ for this ODE. Justify your answer.

(b) (2 points) Find the exponents (roots of the indicial equation) at $x = 0$ for this ODE.

(c) (5 points) Find one non-trivial (i.e., different from $y(x) \equiv 0$) solution of this ODE. Express this solution in the form of infinite series around $x = 0$.

(d) (1 point) Let $y_2(x)$ be any solution of this equation that is linearly independent from the solution you found in (c). What can you say about $\lim_{x \to 0} y_2(x)$? Justify your answer (note: you don't need to find $y_2(x)$ to answer this).

**5.** (a) (5 points) Find the general solution of the system

$$\begin{aligned} x' &= -5x + 2y \\ y' &= -6x + 2y \end{aligned}, \qquad -\infty < t < \infty.$$

(b) (5 points) Classify (by the portrait type and stability type) $(0, 0)$ as a critical point of this system. Make a sketch of the phase portrait.

**6.** Parts (a)–(c) are unrelated.

(a) (2 points) Is the following system linear or non-linear?

$$\begin{aligned} x'(t) &= e^t - x(t) + 2y(t) \\ y'(t) &= x(t) + \sin(t^2)y(t) \end{aligned}, \qquad -\infty < t < \infty.$$

(b) (2 points) Let $A$ be a $2 \times 2$ matrix, $t$ a real number, and $B(t) = \dfrac{d}{dt}\exp(At)$. Find $B(0)$.

(c) (6 points) Suppose $\begin{bmatrix} 2t^2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} t \\ 1/t \end{bmatrix}$ both solve the system $\vec{x}\,'(t) = P(t)\vec{x}(t)$ on $t > 0$ for some $2 \times 2$ matrix $P(t)$. Find the <u>general</u> solution of the system

$$\vec{x}\,'(t) = P(t)\vec{x}(t) + \begin{bmatrix} 0 \\ 1/t \end{bmatrix}, \qquad 0 < t < \infty.$$

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**7.** (a) (2 points) Consider the ODE

$$z''(t) - z'(t) - (z'(t))^3 - z(t) = 0, \qquad -\infty < t < \infty.$$

Reduce this ODE to a system of first order ODEs.

(b) (5 points) Consider the system

$$\begin{aligned} x' &= y \\ y' &= x + y + y^3 \end{aligned}, \qquad -\infty < t < \infty.$$

Find and classify (by the portrait type and stability type) all the critical points of this non-linear system.

(c) (3 points) Prove that the system in (b) has no periodic (non-constant) solutions.

**8.** (a) (2 points) Complete the definition: Let $V$ be a function defined on some domain $D$ containing the origin. Then $V(x, y)$ is called positive definite if...

(b) (8 points) Show that $(0, 0)$ is an unstable critical point of the system

$$\begin{aligned} x' &= 2xy + x^3 \\ y' &= -x^2 + y^5. \end{aligned}$$

Hint: look for $V(x, y) = ax^2 + by^2$.

*GOOD LUCK!!!*