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Matematiska institutionen
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Prov i matematik
Ordinära differentialekvationer I
1MA032, Q2, 2015-12-14

Time: 14.00 – 19.00. Tools allowed: only materials for writing.
Please provide full explanations and calculations in order to get full credit.
The exam consists of **8 problems** *of* **10 points** *each for a total of* **80 points**. *For grades 3,4, and 5, one should obtain 36, 50, and 64 points, respectively.*
Good luck and have fun!

1. (a) (2 points) Complete the following definition: differential equation

$$P(x,y) + Q(x,y)y' = 0$$

is called exact if there exists a function $\psi(x,y)$ such that...

(b) (8 points) Find the general solution of the ODE

$$(xe^{xy} + 2015y)y' = 2016x - ye^{xy}.$$

2. Parts (a)–(d) are unrelated.

(a) (3 points) Find the general solution of the ODE $y'(t) = 1/t$ on the domain $t < 0$.

(b) (2 points) Complete the definition: a collection of functions $\phi_1(t), \ldots, \phi_n(t)$ is called linearly dependent on an interval $\alpha < t < \beta$ if...

(c) (2 points) Rewrite the integral equation $y(t) - \int_2^t (1 + s + e^{y(s)^2})ds = 0$ as an ODE together with an initial condition.

(d) (3 points) Using the Sturm separation theorem, prove that zeros of functions $\sin x + 14\cos x$ and $12\sin x + 2015\cos x$ are distinct and occur alternately (no credit if Sturm theorem is not used).

3. (a) (5 points) Solve the initial value problem

$$\begin{aligned} &y''(x) - 2y'(x) + y(x) = 0, \qquad -\infty < x < \infty, \\ &y(0) = 2015, \quad y'(0) = 2016. \end{aligned}$$

(b) (5 points) Find the general solution of the ODE

$$y''(x) - 2y'(x) + y(x) = e^x, \qquad -\infty < x < \infty.$$

Continuation on the next page

4. Consider the ODE

$$xy'' + y' - y = 0.$$

(a) (2 points) Classify (ordinary/regular singular/irregular singular) the point $x = 0$ for this ODE. Justify your answer.

(b) (2 points) Find the exponents (roots of the indicial equation) at $x = 0$ for this ODE.

(c) (5 points) Find one non-trivial (i.e., different from $y(x) \equiv 0$) solution of this ODE. Express this solution in the form of infinite series around $x = 0$.

(d) (1 point) Let $y_2(x)$ be any solution of this equation that is linearly independent from the solution you found in (c). What can you say about $\lim_{x \to 0} y_2(x)$? Justify your answer (note: you don't need to find $y_2(x)$ to answer this).

5. (a) (5 points) Find the general solution of the system

$$\begin{aligned} x' &= -5x + 2y \\ y' &= -6x + 2y \end{aligned}, \qquad -\infty < t < \infty.$$

(b) (5 points) Classify (by the portrait type and stability type) $(0,0)$ as a critical point of this system. Make a sketch of the phase portrait.

6. Parts (a)–(c) are unrelated.

(a) (2 points) Is the following system linear or non-linear?

$$\begin{aligned} x'(t) &= e^t - x(t) + 2y(t) \\ y'(t) &= x(t) + \sin(t^2)y(t) \end{aligned}, \qquad -\infty < t < \infty.$$

(b) (2 points) Let A be a 2×2 matrix, t a real number, and $B(t) = \dfrac{d}{dt}\exp(At)$. Find $B(0)$.

(c) (6 points) Suppose $\begin{bmatrix} 2t^2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} t \\ 1/t \end{bmatrix}$ both solve the system $\vec{x}\,'(t) = P(t)\vec{x}(t)$ on $t > 0$ for some 2×2 matrix $P(t)$. Find the <u>general</u> solution of the system

$$\vec{x}\,'(t) = P(t)\vec{x}(t) + \begin{bmatrix} 0 \\ 1/t \end{bmatrix}, \qquad 0 < t < \infty.$$

Continuation on the next page

7. (a) (2 points) Consider the ODE

$$z''(t) - z'(t) - (z'(t))^3 - z(t) = 0, \qquad -\infty < t < \infty.$$

Reduce this ODE to a system of first order ODEs.

(b) (5 points) Consider the system

$$\begin{aligned} x' &= y \\ y' &= x + y + y^3 \end{aligned}, \qquad -\infty < t < \infty.$$

Find and classify (by the portrait type and stability type) all the critical points of this non-linear system.

(c) (3 points) Prove that the system in (b) has no non-constant periodic solutions.

8. (a) (2 points) Complete the definition: Let V be a function defined on some domain D containing the origin. Then $V(x, y)$ is called positive definite if...

(b) (8 points) Show that $(0, 0)$ is an unstable critical point of the system

$$\begin{aligned} x' &= 2xy + x^3 \\ y' &= -x^2 + y^5. \end{aligned}$$

Hint: look for $V(x, y) = ax^2 + by^2$.

GOOD LUCK!!!

Solutions to ODE-1 exam, 2015-12-14

Problem 1

(a) Diff. equation $P(x,y) + Q(x,y)\,y' = 0$ is called <u>exact</u> if there exists a function $\psi(x,y)$ such that $\frac{\partial \psi}{\partial x} = P(x,y)$ and $\frac{\partial \psi}{\partial y} = Q(x,y)$

(i.e. if the equation can be rewritten as $\frac{d}{dx}\left(\psi(x, y(x))\right) = 0$

(b) Rewrite our ODE as

$$\underbrace{(y e^{xy} - 2016x)}_{P(x,y)} + \underbrace{(x e^{xy} + 2015y)}_{Q(x,y)}\, y' = 0$$

Check for exactness: $\frac{\partial P}{\partial y} = 1 \cdot e^{xy} + yx e^{xy}$

$\frac{\partial Q}{\partial x} = 1 \cdot e^{xy} + xy e^{xy}$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, we know that the equation is exact.

Let us find ψ:

$$\begin{cases} \frac{\partial \psi}{\partial x} = y e^{xy} - 2016x \\ \frac{\partial \psi}{\partial y} = x e^{xy} + 2015y \end{cases}$$

$\frac{\partial \psi}{\partial x} = y e^{xy} - 2016x \Rightarrow$ integrate with respect to x: $\psi(x,y) = e^{xy} - 1008x^2 + h(y)$ ← arbitrary function of y

$\frac{\partial \psi}{\partial y} = x e^{xy} + 2015y \Rightarrow$ plug in this into the second equation:

$$\cancel{x e^{xy}} - 0 + h'(y) = \cancel{x e^{xy}} + 2015y$$

$$h(y) = \frac{2015}{2} y^2 = 1007.5\, y^2$$ (arbitrary constant for h can be ignored)

So $\psi(x,y) = e^{xy} - 1008x^2 + 1007.5\,y^2$

Thus our ODE becomes

$$\frac{d}{dx}\left(e^{xy} - 1008x^2 + 1007.5\,y^2\right) = 0$$

$\Rightarrow$ general solution is

$$e^{xy} - 1008x^2 + 1007.5\,y^2 = c,$$ where c is an arbitrary constant

Problem 2)

(a) Integrate $y'(t) = \frac{1}{t} \Rightarrow y(t) = \log|t| + c$

Since $t<0$, we have $y(t) = \log(-t) + c$, where c is an arbitrary constant.

(b) A collection of functions $\varphi_1(t), \dots, \varphi_n(t)$ is called <u>linearly dependent on an interval $\alpha < t < \beta$</u> if there exist constants $k_1, \dots, k_n$ such that

$$k_1 \varphi_1(t) + \dots + k_n \varphi_n(t) = 0 \quad \text{for all } t \text{ on } \alpha < t < \beta.$$

(c) $$y(t) = \int_2^t (1 + s + e^{y(s)^2})\, ds$$

Differentiating with respect to t (use Fundamental Theorem of Calculus):

$$y'(t) = 1 + t + e^{y(t)^2}$$

Initial condition: $y(2) = 0$

(d) Note that both functions $\sin x + 14\cos x$ and $12\sin x + 2015\cos x$ solve the differential equation $y''(t) + y(t) = 0$.

Also note that these two functions are linearly independent, since their Wronskian is nonzero:

$$W(x)\Big|_{x=0} = \det \begin{bmatrix} \sin x + 14\cos x & 12\sin x + 2015\cos x \\ \cos x - 14\sin x & 12\cos x - 2015\sin x \end{bmatrix}\Bigg|_{x=0} =$$

$$= \det \begin{bmatrix} 14 & 2015 \\ 1 & 12 \end{bmatrix} = 12 \cdot 14 - 2015 = -1847 \neq 0$$

Thus by the Sturm separation theorem, zeros of $\sin x + 14\cos x$ and $12\sin x + 2015\cos x$ are distinct and occur alternately.

Problem 3

(a) $$y'' - 2y' + y = 0$$
$$y(0) = 2015$$
$$y'(0) = 2016$$

Characteristic equation: $r^2 - 2r + 1 = 0$

$\Rightarrow r_{1,2} = 1$

So general solution is $c_1 e^x + c_2 x e^x$

Plugging in the initial conditions:

$$y(0) = c_1 = 2015$$
$$y'(0) = c_1 e^x + c_2 (e^x + x e^x)\Big|_{x=0} = c_1 + c_2 = 2016$$

So $c_1 = 2015$, $c_2 = 1$, and the solution is

$$2015\, e^x + x e^x$$

(b) $$y''(x) - 2y'(x) + y(x) = e^x$$

Let us use method of undetermined coefficients to find a particular solution of this non-homogeneous equation.

$$Y(x) = A x^2 e^x$$

because $r = 1$ is a double root of the homogeneous equation, see (a)

$$Y'(x) = 2Axe^x + Ax^2 e^x$$
$$Y''(x) = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x$$

So $$Y'' - 2Y' + Y = 2Ae^x + \cancel{4Axe^x} + \cancel{Ax^2e^x} - \cancel{4Axe^x} - \cancel{2Ax^2e^x} + \cancel{Ax^2e^x} =$$
$$= e^x$$

Thus $2A = 1 \Rightarrow A = \frac{1}{2}$

So $Y(x) = \frac{1}{2} x^2 e^x$

General solution of non-homogeneous equation is equal to particular solution plus general solution of homogeneous equation (see part (a)). So:

$$\frac{1}{2} x^2 e^x + c_1 e^x + c_2 x e^x$$ for arbitrary constants c_1 and c_2

$\boxed{\text{Problem 4}}$ $\quad xy'' + y' - y = 0$

(a) Divide by x : $\quad y'' + \underbrace{\frac{1}{x}}_{p(x)} y' - \underbrace{\frac{1}{x}}_{q(x)} y = 0$

$p(x)$ and $q(x)$ are <u>not</u> analytic at $x=0$, so $x=0$ is a singular point.

But $\quad xp(x) = 1$
and
$x^2 q(x) = -x$ are analytic, so $x=0$ is <u>regular singular point</u>

(b) Note that $\quad p_0 = \lim_{x\to 0} xp(x) = 1$

$q_0 = \lim_{x\to 0} x^2 q(x) = 0$, so

indicial equation is

$$r(r-1) + 1\cdot r + 0 = 0$$
$$r^2 - r + r = 0$$
$$\underline{r_{1,2} = 0}$$

(c) By the Frobenius method one solution is of the form

$$y = x^m \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+m} \quad \text{with} \quad a_0 \neq 0$$

$$y' = \sum_{n=0}^{\infty} a_n (n+m) x^{n+m-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+m)(n+m-1) x^{n+m-2} \quad \Rightarrow \quad xy'' = \sum_{n=0}^{\infty} a_n (n+m)(n+m-1) x^{n+m-1}$$

Plug this into the ODE:

$$\sum_{n=0}^{\infty} a_n (n+m)(n+m-1) x^{n+m-1} + \sum_{n=0}^{\infty} a_n (n+m) x^{n+m-1} - \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

or:

$$\sum_{n=-1}^{\infty} a_{n+1} (n+m+1)(n+m) x^{n+m} + \sum_{n=-1}^{\infty} a_{n+1} (n+m+1) x^{n+m} - \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

Now we need to equate all the coefficients near the corresponding powers of x to zero:

Coeff near x^{m-1} (i.e. $n=-1$): $\quad a_0 m(m-1) + a_0 m = 0$

$$\Rightarrow \quad m(m-1) + m = 0 \quad \Rightarrow \quad m = 0$$

Coeff. near x^{n+m} for $n \geq 0$: $\quad a_{n+1}(n+m+1)(n+m) + a_{n+1}(n+m+1) - a_n = 0$

$$\Rightarrow \quad a_{n+1} = \frac{a_n}{(n+m+1)^2} = \frac{a_n}{(n+1)^2} \quad \text{since } m=0.$$

Thus we get $a_1 = \frac{a_0}{1^2} = a_0$

$$a_2 = \frac{a_1}{2^2} = \frac{a_0}{2^2}$$

$$a_3 = \frac{a_2}{3^2} = \frac{a_1}{3^2\,2^2} = \frac{a_0}{(3!)^2}$$

...

$$a_n = \frac{a_{n-1}}{n^2} = \frac{a_{n-2}}{n^2(n-1)^2} = \frac{a_{n-3}}{n^2(n-1)^2(n-2)^2} = \dots = \frac{a_0}{(n!)^2}$$

So we obtained the following solution:

$$y(x) = x^0 \sum_{n=0}^{\infty} \frac{a_0}{(n!)^2} x^n = a_0 \sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^n$$

arbitrary constant

So one non-trivial solution is

$$\sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^n$$

(d) We know from the Frobenius method that if the indicial equation has a double root, the one of the solutions $y_1(x)$ can be found in the Frobenius series form $x^m \sum_{n=0}^{\infty} a_n x^n$ (as we did in (c)), and the other solution will have $y_1(x) \ln x$ as a summand. Therefore $\lim_{x \to 0} y_2(x)$ doesn't exist (it could be $+\infty$ or $-\infty$ depending on a multiplicative constant in front of $y_1(x) \ln x$)

Problem 5

(a) $x' = -5x + 2y$

$y' = -6x + 2y$, i.e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} -5 & 2 \\ -6 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$

Eigenvalues of A: $\det(A - rI) = \det \begin{bmatrix} -5-r & 2 \\ -6 & 2-r \end{bmatrix} =$

$= r^2 + 3r + 2 = 0 \implies r_{1,2} = \dfrac{-3 \pm \sqrt{9-8}}{2} = \dfrac{-3 \pm 1}{2}$

So $r_1 = -1$

$r_2 = -2$

Eigenvectors of A corresponding to $r_1 = -1$:

$$\begin{bmatrix} -5 & 2 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = - \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \implies \begin{cases} -5\xi_1 + 2\xi_2 = -\xi_1 \\ -6\xi_1 + 2\xi_2 = -\xi_2 \end{cases}$$

$$\implies \begin{cases} -4\xi_1 + 2\xi_2 = 0 \\ -6\xi_1 + 3\xi_2 = 0 \end{cases} \implies -2\xi_1 + \xi_2 = 0$$

So $\xi_2 = s$ can be arbitrary, $\xi_1 = \dfrac{\xi_2}{2} = \dfrac{s}{2}$

All eigenvectors are $\begin{bmatrix} s/2 \\ s \end{bmatrix}$, e.g. we can take $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Eigenvectors of A corresponding to $r_2 = -2$:

$$\begin{bmatrix} -5 & 2 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = -2 \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \implies \begin{cases} -5\xi_1 + 2\xi_2 = -2\xi_1 \\ -6\xi_1 + 2\xi_2 = -2\xi_2 \end{cases}$$

$$\implies \begin{cases} -3\xi_1 + 2\xi_2 = 0 \\ -6\xi_1 + 4\xi_2 = 0 \end{cases} \implies \xi_1 = \frac{2}{3}\xi_2$$

So $\xi_2 = s$ can be arbitrary, and $\xi_1 = \frac{2}{3}s$.

All eigenvectors are $\begin{bmatrix} \frac{2}{3}s \\ s \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} s$, e.g. $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

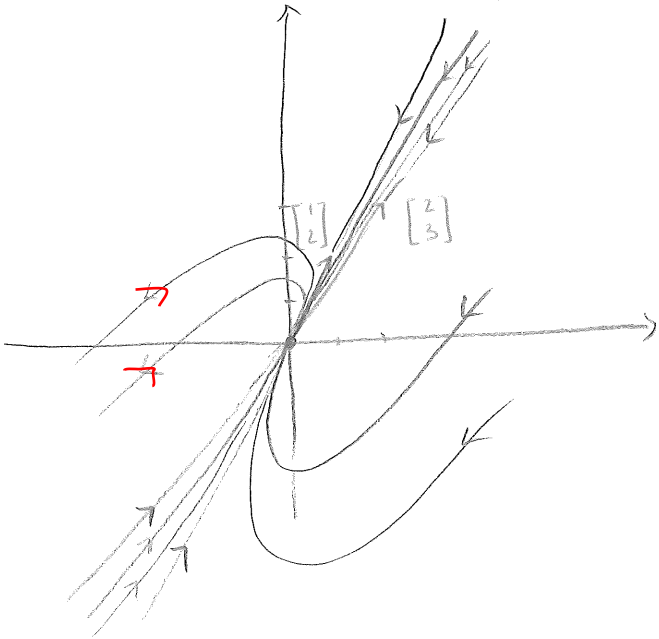
General solution of the system is therefore

$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t}$ where c_1 and c_2 are arbitrary constants.

(b) Because both eigenvalues of A are negative, (0,0) is a nodal sink. In fact eigenvalue do not coincide, so (0,0) is a <u>proper nodal sink</u>.

It is an <u>asymptotically stable</u> critical point since it is a nodal sink (by part (a): all trajectories converge to (0,0) as $t \to +\infty$).

Finally, to sketch a phase portrait, it is useful to sketch eigenvectors $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\3\end{bmatrix}$:



Useful to note that trajectories of $c_1\begin{bmatrix}1\\2\end{bmatrix}e^{-t} + c_2\begin{bmatrix}2\\3\end{bmatrix}e^{-2t}$ become very close (tangential) to the direction of vector $c_1\begin{bmatrix}1\\2\end{bmatrix}$ as $t \to +\infty$ and when $t \to -\infty$, the direction of trajectories start resembling $\pm\begin{bmatrix}2\\3\end{bmatrix}$

Problem 6

(a) System
$$x' = e^t - x + 2y$$
$$y' = x + \sin(t^2)\cdot y \quad \text{is linear}$$

(b) $$B(t) = \frac{d}{dt}\exp(At) \underset{\text{by definition of exp}}{=} \frac{d}{dt}\sum_{n=0}^{\infty}\frac{A^n t^n}{n!} = \sum_{n=1}^{\infty} n\frac{A^n t^{n-1}}{n!} =$$

$$= A\cdot\sum_{n=1}^{\infty}\frac{A^{n-1}t^{n-1}}{(n-1)!} \underset{\text{shift in index}}{=} A\sum_{n=0}^{\infty}\frac{A^n t^n}{n!} = A\exp(At)$$

Thus
$$B(0) = A\exp(0) = A\cdot I = \underline{\underline{A}}$$

(c) We use the method of variation of parameters to find a particular solution of the non-homogeneous system. Note that $\begin{bmatrix}2t^2\\1\end{bmatrix}$ and $\begin{bmatrix}t\\1/t\end{bmatrix}$ are linearly independent since their Wronskian is $\det\begin{bmatrix}2t^2 & t\\ 1 & 1/t\end{bmatrix} = t \neq 0$ on $0<t<\infty$. The method says that we may look for

$$v(t) = \begin{bmatrix}2t^2\\1\end{bmatrix}u_1(t) + \begin{bmatrix}t\\1/t\end{bmatrix}u_2(t)\ , \text{ where functions}$$

u_1 and u_2 satisfy
$$\begin{bmatrix}2t^2 & t\\ 1 & 1/t\end{bmatrix}\begin{bmatrix}u_1'(t)\\u_2'(t)\end{bmatrix} = \begin{bmatrix}0\\1/t\end{bmatrix}$$

By using Cramer's Rule (or solving the linear system in any other way), we get:

$$u_1'(t) = \frac{\det\begin{bmatrix}0 & t\\ 1/t & 1/t\end{bmatrix}}{\det\begin{bmatrix}2t^2 & t\\ 1 & 1/t\end{bmatrix}} = \frac{-1}{t} = -\frac{1}{t} \Rightarrow u_1(t) = -\ln|t| + c_1$$

$$u_2'(t) = \frac{\det\begin{bmatrix}2t^2 & 0\\ 1 & 1/t\end{bmatrix}}{\det\begin{bmatrix}2t^2 & t\\ 1 & 1/t\end{bmatrix}} = \frac{2t}{t} = 2 \Rightarrow u_2(t) = 2t + c_2$$

So $v(t) = \begin{bmatrix}2t^2\\1\end{bmatrix}(-\ln|t|) + \begin{bmatrix}t\\1/t\end{bmatrix}\cdot 2t$ is a particular solution.

General solution is $\boxed{c_1\begin{bmatrix}2t^2\\1\end{bmatrix} + c_2\begin{bmatrix}t\\1/t\end{bmatrix} - \begin{bmatrix}2t^2\\1\end{bmatrix}\ln|t| + \begin{bmatrix}t\\1/t\end{bmatrix}\cdot 2t}$

Problem 7

(a) $z''(t) - z'(t) - (z'(t))^3 - z(t) = 0$

Let $x = z$
$y = z'$

Then $x' = y$
$y' = z'' = z' + (z')^3 + z = y + y^3 + x$

So our system is $\begin{cases} x' = y \\ y' = x + y + y^3 \end{cases}$

(b) Critical points are solutions of $\begin{cases} y = 0 \\ x + y + y^3 = 0 \end{cases}$,

i.e. $x = y = 0$ is the unique critical point

At $(0,0)$ the system is locally-linear since $F(x,y) = y$ and $G(x,y) = x + y + y^3$ have continuous partial derivatives of any order.

The linearized system at $(0,0)$ is

$\begin{cases} x' = y \\ y' = x + y \end{cases}$ i.e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Eigenvalues of $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ are: $\det \begin{bmatrix} -r & 1 \\ 1 & 1-r \end{bmatrix} = r^2 - r - 1 = 0$

$\Rightarrow r_{1,2} = \frac{1 \pm \sqrt{5}}{2}$

We see that r_1 and r_2 are real and of opposite sign. So $(0,0)$ is an (unstable) saddle point of the linear system.

By our "perturbation theorem", $(0,0)$ is also an unstable saddle point of the non-linear system.

(c) By one of the Poincaré-Bendixson theorems, every closed trajectory of a system must enclose at least one critical point, and if it's unique then it can't be a saddle point. Since our unique critical point is a saddle point, we can't have a closed trajectory (i.e. a periodic non-constant solution).

Alternatively: $F_x + G_y = 0 + 1 + 3y^2 > 0$, so another theorem works too.

Problem 8

(a) Let V be a function defined on some domain D containing the origin. Then $V(x,y)$ is called <u>positive definite</u> if $V(0,0)=0$ and $V(x,y)>0$ for every other point (x,y) in D.

(b) Let $V(x,y) = ax^2 + by^2$ for some a, b.

Then $\dot{V} = \frac{\partial V}{\partial x} x' + \frac{\partial V}{\partial y} \cdot y' = 2ax(2xy + x^3) + 2by(-x^2+y^5) =$

$= 4ax^2y + 2ax^4 - 2bx^2y + 2by^6 = (4a-2b)x^2y + 2ax^4 + 2by^6$

This function is positive definite if $4a-2b=0$ and $a>0$, $b>0$

E.g. if $a=1$, $b=2$, then

$V(x,y) = x^2+2y^2$ - positive definite

$\dot{V}(x,y) = 2x^4+4y^6$ - positive definite.

So "energy" is strictly increasing along trajectories of our system, and at $(0,0)$ the energy is minimal.

By the Liapunov Second Method, point $(0,0)$ is <u>unstable</u>.