UPPSALA UNIVERSITET
Matematiska institutionen
Rostyslav Kozhan

Prov i matematik
Ordinära differentialekvationer I
1MA032, 2016-03-23

Time: 8.00 – 13.00. Tools allowed: only materials for writing.
Please provide full explanations and calculations in order to get full credit.
The exam consists of **8 problems** *of* **10 points** *each for a total of* **80 points**. *For grades 3,4, and 5, one should obtain 36, 50, and 64 points, respectively.*
Good luck and have fun!

1. (a) (2 points) Differential equation $\dfrac{d^2y}{dt^2} - ty = e^{2t}\sin t$ is

(i) linear;

(ii) non-linear.

(b) (2 points) Complete the definition: the order of the ordinary differential equation is defined to be equal to

(i) the number of arbitrary constants that appear in the general solution;

(ii) the order of the highest derivative that appears in the equation;

(iii) the number of independent variables that appear in the equation;

(iv) the number of dependent variables that appear in the equation.

(c) (2 point) Complete the definition: the Wronskian of two functions $y_1(t)$ and $y_2(t)$ is defined to be...

(d) (2 points) Suppose $y_1(t)$ and $y_2(t)$ solve the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0,$$

and $y_3(t)$ and $y_4(t)$ solve the non-homogeneous problem

$$y'' + p(t)y' + q(t)y = g(t).$$

Which of the following functions also solve the above homogeneous equation:

(i) $y_1(t) + 2016y_2(t)$ YES NO

(ii) $y_3(t) + y_4(t)$ YES NO

(iii) $17y_1(t) - 13y_2(t) + 3y_3(t) - 3y_4(t)$ YES NO

(iv) $(p(0) + q(0) + g(0))y_1(t)$ YES NO

(e) (2 points) Consider the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$. Complete the definition: a point x_0 is called a singular point of this ODE if...

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2. (a) (2 point) Complete the definition: Suppose $\vec{x}^{\,0}$ is a critical point of a differential system. If for any $\epsilon > 0$ there is a $\delta > 0$ such that every solution $\vec{x} = \vec{\phi}(t)$ which satisfies $||\vec{\phi}(0) - \vec{x}^{\,0}|| < \delta$ must also satisfy $||\vec{\phi}(t) - \vec{x}^{\,0}|| < \varepsilon$ for all $t \geq 0$, then $\vec{x}^{\,0}$ is called

(i) stable;
(ii) asymptotically stable;
(iii) unstable;
(iv) continuous;
(v) unreasonable.

(b) (2 points) Suppose $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and let t be a real number. What is $\exp(At)$?

ignore part (c) as it wasn't part of our course

(c) (3 points) Prove that zeros of functions $\cos(3x)$ and $\sin(3x) + 2016\cos(3x)$ are distinct and occur alternately (Hint: Sturm separation theorem).

(d) (3 points) Find equations of trajectories of the autonomous system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= 2xy \end{aligned} \qquad -\infty < t < \infty.$$

3. (a) (8 points) Find the solution of the initial value problem

$$xy'(x) + 4y(x) = x^2, \quad y(-1) = 7/6$$

(b) (2 points) Determine the interval of existence of this solution.

4. (a) (5 points) Find the general solution of the ODE

$$y''(x) - 2y'(x) + 5y(x) = 0, \qquad -\infty < x < \infty$$

(b) (5 points) Find the general solution of the ODE

$$y''(x) - 2y'(x) + 5y(x) = 3\cos x, \qquad -\infty < x < \infty,$$

5. (10 points) Consider the initial value problem

$$\begin{aligned} &(x+2)y''(x) - y(x) = x^2 \\ &y(-1) = -1 \\ &y'(-1) = 2 \end{aligned}$$

Seek power series solutions of this ODE about $x_0 = -1$. Find the first four coefficients explicitly, and find the recurrence relation for the rest of the coefficients.

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6. (a) (5 points) Find the general solution of the system

$$\begin{aligned} x' &= x + y \\ y' &= 2x \end{aligned} \qquad -\infty < t < \infty.$$

(b) (5 points) Classify (by the portrait type and stability type) $(0,0)$ as a critical point of this system. Make a sketch of the phase portrait.

7. (a) (7 points) Consider the system

$$\begin{aligned} x' &= x^2 + y \\ y' &= x - y \end{aligned} \qquad -\infty < t < \infty.$$

Find and classify (by the portrait type and stability type) all the critical points of this non-linear system. Justify your conclusions carefully.

(b) (3 points) Does there exist a periodic solution $(x(t), y(t))$ of the system that satisfies $x(t) > 0$ and $y(t) > 0$ for all t?

8. (a) (2 points) Complete the definition: Let V be a function defined on some domain D containing the origin. Then $V(x,y)$ is called negative definite if...

(b) (8 points) Show that $(0,0)$ is an asymptotically stable critical point of the system

$$\begin{aligned} x' &= -x^3 + 2xy \\ y' &= -x^6 - 3y. \end{aligned}$$

(try to) HAVE FUN and GOOD LUCK! :)