Problem 1

(a) (i)

(b) (ii)

(c) $\det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$.

(d) (i) YES

(ii) NO

(iii) YES

(iv) YES

(e) ... if $P(x_0) = 0$ but $Q(x_0) \neq 0$ or $R(x_0) \neq 0$

Problem 2

(a) (i)

(b) $\exp(At) := \sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 1^n & 0 \\ 0 & 2^n \end{pmatrix} t^n =$

$= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} 1^n t^n & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{1}{n!} 2^n t^n \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$

(c) Both $\cos(3x)$ and $\sin(3x) + 2016\cos(3x)$ solve the ODE $y''(x) + \frac{1}{9} y(x) = 0$

Moreover, they are linearly independent since their Wronskian is non-zero (check this!)

By Sturm separation theorem, zeros occur alternately

(d) $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = 2xy \end{cases} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 2x \Rightarrow dy = 2x\,dx$

$\Rightarrow \boxed{y = x^2 + c}$

Problem 3

(a) $x y' + 4y = x^2$

$$y' + \frac{4}{x} y = x$$

Integrating factor : $e^{\int \frac{4}{x} dx} = e^{4 \ln x} = x^4$

$$x^4 y' + 4x^3 y = x^5$$

$$\frac{d}{dx}\left(x^4 y(x) \right) = x^5$$

$$x^4 y(x) = \frac{x^6}{6} + c$$

$$y(x) = \frac{x^2}{6} + \frac{c}{x^4}$$

Initial condition $y(-1) = 7/6 \Rightarrow c = 1$

So $y(x) = \frac{x^2}{6} + \frac{1}{x^4}$

(b) Our solution is not defined at $x = 0$

It is defined at $x = -1$

So interval of existence is $-\infty < x < 0$, i.e. $(-\infty ; 0)$

Problem 4

(a) $y''(x) - 2y'(x) + 5y(x) = 0$

Charact. equation : $\lambda^2 - 2\lambda + 5 = 0 \Rightarrow \lambda = 1 \pm 2i$

So general solution is $y(x) = c_1 e^x \cos 2x + c_2 e^x \sin 2x$

(b) Let $y_p = A \cos x + B \sin x$ (method of undetermined coefficients)

Then $y_p' = -A \sin x + B \cos x$
$y_p'' = -A \cos x - B \sin x$

Then plug this into $y'' - 2y' + 5y = 3 \cos x$ and equate coefficients near cosines and sines

One gets $A = \frac{3}{5}$, $B = -\frac{3}{10}$

So particular solution is $y_p = \frac{3}{5} \cos x - \frac{3}{10} \sin x$

So general solution is

$$\frac{3}{5} \cos x - \frac{3}{10} \sin x + c_1 e^x \cos 2x + c_2 e^x \sin 2x$$

Problem 5

$$\begin{cases} (x+2)\, y''(x) - y(x) = x^2 \\ y(-1) = -1 \\ y'(-1) = 2 \end{cases}$$

Let us "shift" our problem to the origin:

$x+1 = t$. Then $\frac{d}{dt} y(t) = \frac{d}{dx} y(x)$, $\frac{d^2}{dt^2} y(t) = \frac{d^2}{dx^2} y(x)$, so

$$\begin{cases} (t+1)\, y''(t) - y(t) = (t-1)^2 = t^2 - 2t + 1 \\ y(0) = -1 \\ y'(0) = 2 \end{cases}$$

$t=0$ is an ordinary point

Let $y(t) = \sum_{n=0}^{\infty} c_n t^n$ with $c_0 = -1$, $c_1 = 2$

$y'(t) = \sum_{n=0}^{\infty} n c_n t^{n-1}$

$y''(t) = \sum_{n=0}^{\infty} n(n-1) c_n t^{n-2}$

$$\Rightarrow \sum_{n=0}^{\infty} n(n-1) c_n t^{n-1} + \sum_{n=0}^{\infty} n(n-1) c_n t^{n-2} - \sum_{n=0}^{\infty} c_n t^n = t^2 - 2t + 1$$

$$\Rightarrow \sum_{n=1}^{\infty} (n+1) n\, c_{n+1} t^n + \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} t^n - \sum_{n=0}^{\infty} c_n t^n = t^2 - 2t + 1$$

Equate powers of t^0: $2c_2 - c_1 = 1 \Rightarrow c_2 = 0$

powers of t^1: $2c_2 + 6c_3 - c_1 = -2 \Rightarrow c_3 = 0$

powers of t^2: $6c_3 + 12c_4 - c_2 = 1 \Rightarrow c_4 = 1/12$

powers of t^n $(n \geq 3)$: $(n+1) n\, c_{n+1} + (n+2)(n+1) c_{n+2} - c_n = 0$

$$\Rightarrow c_{n+2} = \frac{c_n - n(n+1) c_{n+1}}{(n+1)(n+2)}$$

Problem 6

(a) $$A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = -1$

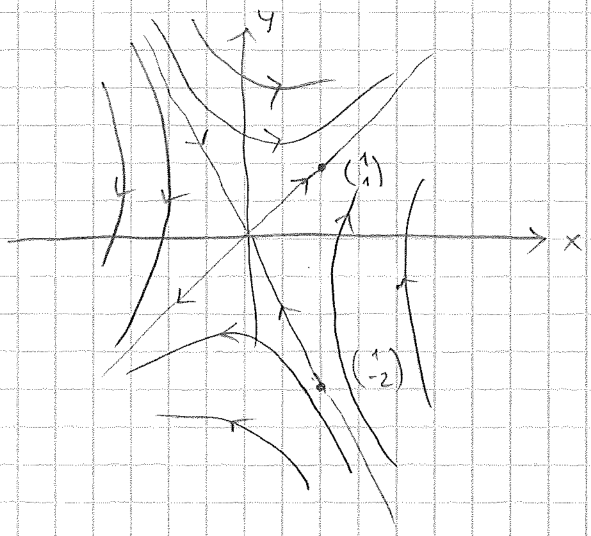
An eigenvector for $\lambda_1 = 2$: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

An eigenvector for $\lambda_2 = -1$: $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

So general solution is $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(b) Since eigenvalues have opposite signs, $(0,0)$ is an unstable saddle point

Sketch:



Problem 7

(a) Critical points: $\begin{cases} x^2 + y = 0 \\ x - y = 0 \end{cases} \Rightarrow \begin{array}{l} (x,y) = (0,0) \\ \text{or } (x,y) = (-1,-1) \end{array}$

Both functions x^2+y and $x-y$ are polynomials $\Rightarrow$ have continuous derivatives of any order $\Rightarrow$ our system is locally-linear at every point

Linearization around $(0,0)$: $\begin{cases} x' = y \\ y' = x - y \end{cases}$ has $A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ with eigenvalues $\frac{-1 \pm \sqrt{5}}{2}$ of opposite signs $\Rightarrow$ $(0,0)$ is unstable saddle point for the linearization system, as well as for our original system

Linearization around $(1,1)$ has matrix $A = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$ with eigenvalues $\frac{-3\pm\sqrt{5}}{2}$ which are both negative. So $(0,0)$ is asymptotically stable node (sink) of our linearized system, which implies that $(-1,-1)$ is an asymptoticall stable sink of our original system.

(b) By Poincare-Bendixson theory, every periodic solution must enclose at least one critical point of the system. There are no critical points of the system in the region $x>0,\ y>0$, so there cannot be any periodic solutions.

Problem 8

(a) ... if $V(0,0)=0$
and $V(x,y)<0$ for all $(x,y)\neq(0,0)$ on D.

(b) Let $V(x,y) = ax^n + by^m$

$$\dot{V} = nax^{n-1}(2xy - x^3) + mby^{m-1}(-3y - x^6)$$

$$= -3mb\,y^m - nax^{n+2} + \underbrace{2n\,ax^n y - mby^{m-1}x^6}_{\text{want to cancel out}}$$

Let $n=6$
$m=2$
$a=1$
$b=6$

Then $V(x,y) = x^6 + 6y^2$ is positive definite

$\dot{V}(x,y) = -6x^8 - 36y^2$ is negative definite

So by Liapunov theorem, $(0,0)$ is an asymtotically stable critical point.