UPPSALA UNIVERSITET
Matematiska institutionen
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Prov i matematik
Ordinära differentialekvationer I
1MA032, 2016-12-16

Time: 8.00 – 13.00. Tools allowed: only materials for writing.
Please provide full explanations and calculations in order to get full credit, except for the Problem 1.
The exam consists of **8 problems** *of 16, 12, 12, 12, 12, 12, 12, 12 points, respectively, for a total of* **100 points**. *For grades 3, 4, and 5, one should obtain 45, 63, and 80 points, respectively.*

1. (a) (2 points) State what the order of the following ODE is:

$$y''(x) + 2x^{-5}y(x) = (\sin x)^4 y'''(x)$$

(b) (2 points) The ODE in part (a) is:

(i) linear.

(ii) non-linear.

(c) (2 points) Complete the definition: an ODE $P(x,y)y' + Q(x,y) = 0$ is called *exact* if... . . . .

(d) (2 points) Write the general solution to the Euler equation

$$x^2y''(x) + 3xy'(x) - 3y(x) = 0$$

(e) (2 points) Complete the following definition: x_0 is a *regular singular point* of the ODE

$$y'' + p(x)y' + q(x)y = 0$$

if . . . .

(f) (2 points) Write the *first* order *system* of ODE's that is equivalent to the ODE

$$y'''(t) - y(t)^2 \sin t = t^2$$

(g) (2 points) Complete the definition: Let V be a function defined on some domain D containing the origin. Then $V(x,y)$ is called *negative definite* if . . . .

(h) (2 points) Complete the statement of the Liapunov theorem: Suppose the autonomous system

$$\begin{aligned} x' &= F(x,y) \\ y' &= G(x,y) \end{aligned} \qquad -\infty < t < \infty.$$

has an isolated critical point at $(0,0)$. If there exists a function $V(x,y)$ that is continuous and with continuous partial derivatives such that . . . .
. . . ., then $(0,0)$ is a *stable* critical point.

Continuation on the next page

2. (a) (8 points) Find the general solution of the ODE,

$$3xy^2y'(x) = 1 + x\cos x$$

(b) (1 point) What is the solution in part (a) when

$$y(\pi/2) = 1?$$

(c) (1 point) What is the solution in part (a) when

$$y(-\pi/2) = -1?$$

(d) (2 point) What is the largest possible *interval* of the solution from part (c)?

3. (a) (2 points) Verify that $y_1(x) = e^x$ is a particular solution of the ODE

$$(\sin x)y''(x) + (-2\sin x - \cos x)y'(x) + (\sin x + \cos x)y(x) = 0$$

(b) (10 points) Find the general solution of the ODE in part (a).

4. (a) (5 points) Find the general solution of the ODE

$$y''(t) + y'(t) = 0$$

(b) (7 points) Find the general solution of the ODE

$$y''(t) + y'(t) = -3te^{-t}$$

5. Consider the ODE

$$y''(x) - y(x)e^x = 0$$

(a) (8 points) Find the first five terms (i.e. terms up to x^4) of the general solution of this ODE in the form of power series about the origin.

(b) (4 points) Find the first three nonzero terms in two *linearly independent* particular solutions of your choice. Justify the linear independence of these solutions.

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6. (a) (7 points) Find the general solution of the system

$$\begin{aligned} x' &= 2x - y \\ y' &= 2x \end{aligned} \qquad -\infty < t < \infty.$$

(b) (2 points) Classify (by the portrait type and stability type) $(0,0)$ as a critical point of this system.

(c) (3 points) Make a sketch of the phase portrait.

7. (a) (9 points) Consider the system

$$\begin{aligned} x' &= -x^2 + y^2 \\ y' &= 1 - 2x \end{aligned} \qquad -\infty < t < \infty.$$

Find and classify (by the portrait type and stability type) all the critical points of this non-linear system. Justify your conclusions carefully.

(b) (3 points) Does there exist a periodic solution $(x(t), y(t))$ of the system that satisfies $x(t) < 0$ for all t? Explain (just an answer is not enough).

8. (a) (1 point) Verify (directly) that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} t + 1/2 \\ 2t \end{pmatrix}$ both are a particular solution of the system of ODE's

$$\begin{aligned} x' &= 2x - y \\ y' &= 4x - 2y \end{aligned} \qquad -\infty < t < \infty.$$

(b) (2 points) Check linear independence of the two solutions in (a), and write the general solution of this system of ODE's.

(c) (8 points) Use method of variation of parameters to find a *particular* solution of the system of ODE's

$$\begin{aligned} x' &= 2x - y + \tfrac{1}{t^2} \\ y' &= 4x - 2y + \tfrac{1}{t^2} \end{aligned} \qquad -\infty < t < \infty.$$

(d) (1 point) Find the *general* solution of the system in part (c).

(try to) HAVE FUN and GOOD LUCK! :)