# Answers to 2016-12-16 Exam in ODE 1

**1** (a) Order is 3

(b) Linear

(c) ... if there exists a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = Q(x,y) \quad , \quad \frac{\partial \psi}{\partial y} = P(x,y)$$

(d) Indicial equation is $r(r-1) + 3r - 3 = 0$

$$(r-1)(r+3) = 0$$

So the general solution is $c_1 |x| + c_2 |x|^{-3}$

(e) ... if $x_0$ is not an ordinary point (that is, $p(x)$ or $q(x)$ are not analytic at $x_0$),

<u>and</u> both $(x-x_0)\,p(x)$ and $(x-x_0)^2 q(x)$ are analytic at $x_0$.

(f) taking $x_1(t) = y(t)$, $x_2(t) = y'(t)$, $x_3(t) = y''(t)$, we get

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_1^2 \sin t + t^2 \end{cases}$$

(g) ... if $V(0,0) = 0$

and $V(x,y) < 0$ for all $(x,y)$ in $D$ except $(0,0)$

(h) ... such that $V(x,y)$ is positive definite and

$\dot{V}(x,y) = \frac{\partial V}{\partial x} F + \frac{\partial V}{\partial y} G$ is negative semi-definite.

$\boxed{2}$ (a) $3y^2\,dy = \left(\frac{1}{x} + \cos x\right) dx$

$$y^3 = \ln|x| + \sin x + c$$

$$y = \sqrt{\ln|x| + \sin x + c}$$

(b) $y(\pi/2) = 1 \implies c = -\ln\left(\frac{\pi}{2}\right)$, i.e.

$$y = \sqrt{\ln|x| + \sin x - \ln\frac{\pi}{2}}$$

(c) Same as (b)

(d) Largest interval is $(-\infty, 0)$ since at $x = 0$, the solution doesn't work

$\boxed{3}$ (a) Straightforward

(b) Use method of reduction of order:

$$y(x) = v(x)\, y_1(x) = v(x)\, e^x$$

$\implies$ plug-in into our ODE; we get

$$\sin x \cdot v'' - \cos x \cdot v' = 0$$

Let $v' = w$:

$\sin x \; w' - \cos x \cdot w = 0 \implies$ for example $w(x) = \sin x$

$\implies$ for example $v(x) = -\cos x$

So $y_2(x) = -e^x \cos x$ is a second solution

It's linearly independent with $y_1(x)$ since $W[y_1, y_2] = \det \begin{bmatrix} e^x & -e^x\cos x \\ e^x & -e^x\cos x + e^x \sin x \end{bmatrix}$

$\neq 0$ if $x \neq k\pi$

So general solution is

$$c_1 e^x + c_2 e^x \cos x$$

**4** (a) Characteristic eq: $r^2 + r = 0$
$r(r+1) = 0$

So gen. solution is $y(t) = c_1 + c_2 e^{-t}$

(b) Particular solution $Y(t) = (At + B)\,t\,e^{-t}$

because $e^{-t}$ solved homogeneous equation

$\Rightarrow$ inserting into ODE produces $A = 3/2$, $B = 3$

So $\left(\frac{3}{2}t^2 + 3t\right)e^{-t}$ is a particular solution

So general solution is $c_1 + c_2 e^{-t} + \left(\frac{3}{2}t^2 + 3t\right)e^{-t}$

**5** (a) $y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$ ← higher order terms

So $y'' - y \cdot e^x = 0$ becomes

$$\left(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \dots\right) +$$
$$-\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\right) = 0$$

$$\Rightarrow \underline{2a_2} + \underline{\underline{6a_3 x}} + \underline{\underline{12a_4 x^2}} + 20a_5 x^3 + 30a_6 x^4 + \dots +$$
$$- \underline{a_0} - \underline{\underline{a_1 x}} - \underline{\underline{a_2 x^2}} - a_3 x^3 - a_4 x^4 - \underline{\underline{a_0 x}} - \underline{\underline{a_1 x^2}} - a_2 x^3 - a_3 x^4 - \underline{\underline{\frac{a_0}{2} x^2}} - \frac{a_1}{2} x^3 - \frac{a_2}{2} x^4 +$$
$$- \frac{a_0}{6} x^3 - \frac{a_1}{6} x^4 - \frac{a_0}{24} x^4 + \dots = 0$$

Coefficient near $x^0$: $2a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = \frac{a_0}{2}$

$x^1$: $6a_3 - a_1 - a_0 = 0 \quad \Rightarrow \quad a_3 = \frac{a_0}{6} + \frac{a_1}{6}$

$x^2$: $12a_4 - a_2 - a_1 - \frac{a_0}{2} = 0 \quad \Rightarrow \quad a_4 = \frac{a_0}{24} + \frac{a_1}{12} + \frac{a_2}{12} = \frac{a_0}{12} + \frac{a_1}{12}$

So general solution is $y(x) = a_0 + a_1 x + \frac{a_0}{2}x^2 + \left(\frac{a_0}{6} + \frac{a_1}{6}\right)x^3 +$
$+ \left(\frac{a_0}{12} + \frac{a_1}{12}\right)x^4 + \dots$

(b) $y(x) = a_0\underbrace{\left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)}_{y_1(x)} + a_1\underbrace{\left(x + \frac{x^3}{6} + \frac{x^4}{12} + \dots\right)}_{y_2(x)}$ ← linearly indep. since $W[y_1, y_2] \neq 0$

**6** (a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$

eigenvalues of $A$: $\det \begin{pmatrix} 2-\lambda & -1 \\ 2 & -\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 2 = 0$

$\lambda = 1 \pm i$

eigenvector of $A$ corresponding to $1+i$:

$\begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (1+i)x_1 \\ (1+i)x_2 \end{pmatrix} \Rightarrow \begin{matrix} 2x_1 - x_2 = (1+i)x_1 & \Rightarrow & x_2 = (1-i)x_1 \\ 2x_1 = (1+i)x_2 & \Rightarrow & x_2 = \frac{2}{1+i} x_1 \end{matrix}$

$\Rightarrow$ e.g. $x_1 = 1$
$x_2 = 1-i$

So a complex solution is $\begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1+i)t} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^t (\cos t + i \sin t)$

$= \begin{pmatrix} e^t \cos t + i e^t \sin t \\ e^t \cos t + i e^t \sin t - i e^t \cos t + e^t \sin t \end{pmatrix} = \begin{pmatrix} e^t \cos t \\ e^t \cos t + e^t \sin t \end{pmatrix} + i \begin{pmatrix} e^t \sin t \\ e^t \sin t - e^t \cos t \end{pmatrix}$

Taking real and imaginary parts produces two real solutions.

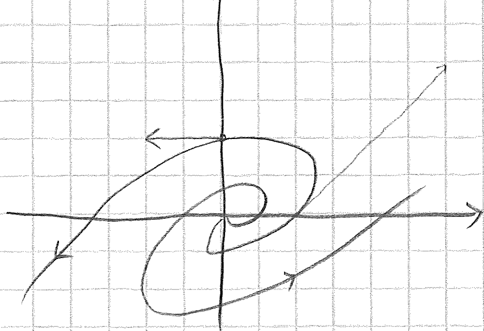
So general solution is

$c_1 e^t \begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}$

(b) Since two complex eigenvalues with positive real parts $\Rightarrow$

$\Rightarrow$ portrait type is spiral source which is unstable

(c)



At point $x=1, y=0$:

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

At point $x=0, y=1$: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

So the spiral is counter-clockw.

$\boxed{8}$ (a) Immediately

(b) Linear independence:

Wronskian $W = \det \begin{bmatrix} 1 & t+\frac{1}{2} \\ 2 & 2t \end{bmatrix} = 2t - 2t - 1 = -1 \neq 0$

General solution: $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} t+1/2 \\ 2t \end{bmatrix}$

(c) $\Psi(t) = \begin{bmatrix} 1 & t+\frac{1}{2} \\ 2 & 2t \end{bmatrix}$

Particular solution: $Y(t) = u_1(t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + u_2(t) \begin{bmatrix} t+\frac{1}{2} \\ 2t \end{bmatrix}$, where

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \Psi(t)^{-1} \begin{bmatrix} 1/t^2 \\ 1/t^2 \end{bmatrix} = \begin{bmatrix} -2t & t+1/2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1/t^2 \\ 1/t^2 \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{2}{t} + \frac{1}{t} + \frac{1}{2t^2} \\ \frac{2}{t^2} - \frac{1}{t^2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{t} + \frac{1}{2t^2} \\ 1/t^2 \end{bmatrix}$$

$$\Rightarrow u_1(t) = \int \left(-\frac{1}{t} + \frac{1}{2t^2}\right) dt = -\ln|t| - \frac{1}{2t} + c_1$$

$$u_2(t) = \int \frac{1}{t^2}\, dt = -\frac{1}{t} + c_2$$

So particular solution, e.g., $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(-\ln|t| - \frac{1}{2t}\right) + \begin{bmatrix} t+\frac{1}{2} \\ 2t \end{bmatrix} \left(-\frac{1}{t}\right)$

$$= -\ln|t| \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{t} \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = -\ln|t| \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(d) General solution is

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} t+1/2 \\ 2t \end{bmatrix} - \ln|t| \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

7 (a) Critical points: $\begin{cases} -x^2+y^2=0 \\ 1-2x=0 \end{cases} \Rightarrow \begin{matrix} x=\frac{1}{2} \\ y=\pm x=\pm\frac{1}{2} \end{matrix}$

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So two critical points $(\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},-\frac{1}{2})$

Note that our ODE system is locally-linear around every point since both $F(x,y)=-x^2+y^2$ and $G(x,y)=1-2x$ are twice continuously differentiable at any point.

Jacobian matrix: $\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} -2x & 2y \\ -2 & 0 \end{pmatrix}$

At $(\frac{1}{2},\frac{1}{2})$: $A=\begin{pmatrix} -1 & 1 \\ -2 & 0 \end{pmatrix} \Rightarrow$ eigenvalues $\frac{-1\pm\sqrt{7}i}{2}$

$\Rightarrow$ stable spiral sink for both linearized and the non-linear systems

At $(\frac{1}{2},-\frac{1}{2})$: $A=\begin{pmatrix} -1 & -1 \\ -2 & 0 \end{pmatrix} \Rightarrow$ eigenvalues $-2, 1$

$\Rightarrow$ unstable saddle point for both linearized and non-linear systems

(b) $F_x+G_y=-2x$ has the same sign in the domain $x<0$, so no periodic non-constant solutions

There are also no constant solutions in the domain $x<0$, as we saw in part (a)