UPPSALA UNIVERSITET
Matematiska institutionen
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Prov i matematik
Ordinära differentialekvationer I
1MA032, 2017-04-21

Time: 8:00—13:00. Tools allowed: only materials for writing.
Please provide full explanations and calculations in order to get full credit, except for the Problem 1.
The exam consists of **8 problems** *of 10 points each, for a total of* **80 points**. *For grades 3, 4, and 5, one should obtain 36, 50, and 64 points, respectively.*

1. (a) (3 points) Differential equation $\frac{d^4y}{dt^4} + t^6\frac{dy}{dt} = -t^8y(t)$ is

(i) linear homogenous;
(ii) linear non-homogenous;
(iii) non-linear.

(b) (4 points) Suppose $y_1(t)$ and $y_2(t)$ solve the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0,$$

and $y_3(t)$ and $y_4(t)$ solve the non-homogeneous problem

$$y'' + p(t)y' + q(t)y = g(t).$$

Which of the following functions also solve the same non-homogeneous equation above?

(i) 0	YES	NO
(ii) $2017y_2(t) - 2016y_1(t)$	YES	NO
(iii) $2017y_3(t) - 2016y_1(t)$	YES	NO
(iv) $2017y_4(t) - 2016y_3(t) + 2015y_2(t) - 2014y_1(t)$	YES	NO

(c) (3 points) Rewrite the integral equation

$$y(t) = \int_{-1}^{t} s^2 y(s)\, ds + 3$$

as an ODE together with an initial condition.

2. (10 points) Find the general solution of the ODE,

$$(3xy + y^2) + (x^2 + xy)y'(x) = 0$$

(Hint: multiply the equation by a function $\mu(x)$ of x only, to make this equation exact)

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3. (a) (4 points) Find the general solution of the ODE

$$y''(t) - 2y'(t) + y(t) = 0$$

(b) (6 points) Find a particular solution of the ODE

$$y''(t) - 2y'(t) + y(t) = \frac{e^t}{t} \qquad (0 < t < \infty)$$

4. (a) (4 points) Find the general solution of the ODE

$$y'''(t) - y'(t) = 0$$

(b) (5 points) Find a <u>particular</u> solution of the ODE

$$y'''(t) - y'(t) = t$$

(c) (1 point) Find the <u>general</u> solution of the ODE in part (b)

5. Consider the ODE

$$x^2y''(x) + xy'(x) + (x^2 - \frac{1}{4})y(x) = 0$$

(a) (2 points) Is $x = 0$ an ordinary, regular singular, or an irregular singular point? Briefly explain.

(b) (8 points) Depending on your answer in (a), we should seek power or Frobenius solutions around $x = 0$ of this ODE. Find the first two <u>non-zero</u> terms of a particular solution of this ODE.

6. (a) (6 points) Find the general solution of the system

$$\begin{aligned} x' &= x + 3y \\ y' &= x - y \end{aligned} \qquad -\infty < t < \infty.$$

(b) (2 points) Classify (by the portrait type and stability type) $(0, 0)$ as a critical point of this system.

(c) (2 points) Make a sketch of the phase portrait.

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7. Consider the system

$$\begin{aligned} x' &= 3x - 2y - x^2 \\ y' &= x + 21y^4 + x^{2017} \end{aligned} \qquad -\infty < t < \infty.$$

(a) (1 point) Verify that $(0,0)$ is a critical point.

(b) (1 point) Show/explain that the system is locally linear at $(0,0)$.

(c) (8 points) Classify (by the portrait type and stability type) $(0,0)$ as the critical point of this non-linear system. Justify your conclusions carefully.

8. (a) (1 point) Find all the critical points of the system

$$\begin{aligned} x' &= y \\ y' &= -x^3 - y^3 \end{aligned} \qquad -\infty < t < \infty.$$

(b) (9 points) Prove that the point(s) you found in part (a) is/are stable. (Hint: look for $V(x,y) = ax^k + by^m$)

(try to) HAVE FUN and GOOD LUCK! :)