UPPSALA UNIVERSITET
Matematiska institutionen
Rostyslav Kozhan

Prov i matematik
Ordinära differentialekvationer I
1MA032, 2018-01-05

Time: 14.00 – 19.00. Tools allowed: only materials for writing.
Please provide full explanations and calculations in order to get full credit, except for the Problem 1.
The exam consists of **8 problems** *of 16, 12, 12, 12, 12, 12, 12, 12 points, respectively, for a total of* **100 points**. *For grades 3, 4, and 5, one should obtain 45, 63, and 80 points, respectively.*

1. For this problem only, no explanations are required – just the correct answer is enough.

(a) (2 points) TRUE or FALSE: The ODE

$$e^x\left(y-\frac{d^2y}{dx^2}\right)=y\cos(x^2)$$

is

(i) linear.
(ii) non-linear.

(b) (2 points) TRUE or FALSE: for any constants a, b, c, all solutions of the equation $ay''(x)+by'(x)+cy(x)=0$ are of the form e^{rt}, where r is some real number.

(c) (2 points) TRUE or FALSE: function $y(x)\equiv 2$ is the *unique* solution of the initial value problem

$$\begin{aligned}&e^x y''(x)-(\sin x)y'(x)+y(x)=2\\&y(0)=2\\&y'(0)=0\end{aligned}$$

(d) (2 points) Suppose $f(x)=\sum_{n=0}^{\infty}a_nx^n$ and $g(x)=\sum_{n=0}^{\infty}b_nx^n$ with positive radius of convergence. Let $f(x)g(x)=\sum_{n=0}^{\infty}c_nx^n$. What is c_5 in terms of a_n's and b_n's?

(e) (2 points) Complete the following definition: x_0 is an *ordinary point* of the ODE

$$y''+p(x)y'+q(x)y=0$$

if ......

Continuation on the next page

(f) (2 points) Integral equation

$$\int_1^t e^{y(s)^2+s^2}\,ds + 2018 = y(t)$$

is equivalent to the following ODE with the initial value condition: ......

(g) (2 points) Suppose $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ has only one eigenvalue $\lambda = -2018$ of (algebraic) multiplicity 2. What can be said about the phase portrait of the system

$$\begin{aligned} x' &= a_{11}x + a_{12}y \\ y' &= a_{21}x + a_{22}y \end{aligned} \qquad -\infty < t < \infty.$$

(state the name of the portrait(s) or make sketch(es))

(h) (2 points) Suppose $(0,0)$ is a isolated critical point of the locally-linear autonomous system

$$\begin{aligned} x' &= F(x,y) \\ y' &= G(x,y) \end{aligned} \qquad -\infty < t < \infty.$$

Suppose eigenvalues of $\begin{pmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{pmatrix}$ are $\pm 2018i$. What are the possible phase portraits of our locally-linear system at $(0,0)$? (state the name of the portrait(s) or make sketch(es))

2. Consider the ODE

$$\tfrac{1}{2}y'(x) = e^{\sin x}\sin(2x) + y(x)\cos x$$

Note: you may still solve (b)–(c) even if you don't solve (a).

(a) (8 points) Find the general solution of this ODE.

(b) (2 points) How many solutions does this ODE have that satisfy the condition $y(\frac{\pi}{2}) = 2018$? Briefly explain your answer.

(c) (2 points) How many solutions does this ODE have that satisfy the condition $y'(\frac{\pi}{2}) = 2018$? Briefly explain your answer.

3. (a) (6 points) Find the general solution of the ODE

$$4y''(t) + 4y'(t) + y(t) = 0$$

(b) (6 points) Find the general solution of the ODE

$$4y''(t) + 4y'(t) + y(t) = e^{-t/2} + e^{t/2}$$

Continuation on the next page

4. (a) (4 points) Find the general solution of the Euler's equation

$$2x^2y''(x) + xy'(x) - 3y(x) = 0$$

(b) (8 points) Find one particular solution of the ODE

$$2x^2y''(x) + (x + x^3)y'(x) - 3y(x) = 0$$

in the form of some infinite (power or Frobenius, whichever is applicable) series around 0 (find the first few coefficients and the recurrence relation for the rest of them).

5. (a) (6 points) Find the general solution of the system

$$\begin{aligned} x' &= 2x - y \\ y' &= -x + 2y \end{aligned} \qquad -\infty < t < \infty.$$

(b) (2 points) Classify (by the portrait type and stability type) $(0,0)$ as a critical point of this system.

(c) (4 points) Make a sketch of the phase portrait. Make sure the eigenvectors are sketched on the same picture. Make sure the behaviour around $(0,0)$ is clear from your picture (which line are the trajectories tangent to?)

6. (a) (4 points) *Method of undetermined coefficients* tells us that a particular solution of the non-homogeneous system of ODE's (compare with Problem 5)

$$\begin{aligned} x' &= 2x - y \\ y' &= -x + 2y + \sin 3t \end{aligned}$$

should be in the following form: $\vec{Y}_1(t) = \ldots\ldots$ (just an answer is enough; leave the coefficients undetermined; you do not need to compute $\vec{Y}_1$ here).

(b) (4 points) *Method of undetermined coefficients* tells us that a particular solution of the non-homogeneous system of ODE's (compare with Problem 5)

$$\begin{aligned} x' &= 2x - y - t^2e^{3t} \\ y' &= -x + 2y + e^{3t} \end{aligned}$$

should be in the following form: $\vec{Y}_2(t) = \ldots\ldots$ (just an answer is enough; leave the coefficients undetermined; you do not need to compute $\vec{Y}_2$ here).

(c) (4 points) Suppose $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ from (a) and (b) are given to you. What is the *general* solution of the system of ODE's

$$\begin{aligned} x' &= 2x - y - t^2e^{3t} \\ y' &= -x + 2y - \sin 3t + e^{3t} \end{aligned}$$

Continuation on the next page

7. Consider the system

$$\begin{aligned} x' &= x(3-x-2y) \\ y' &= y(4-2x-2y) \end{aligned} \qquad -\infty < t < \infty.$$

(a) (3 points) Find all the critical points of this non-linear system.

(b) (6 points) Classify (by the portrait type and stability type) each of the critical points of this non-linear system. Justify your conclusions carefully.

(c) (3 points) Show that there cannot exist a periodic solution $(x(t), y(t))$ of this system that satisfies $x(t) > 0, y(t) > 0$ for all t.

8. (a) (3 points) Complete the statement in the Second Liapunov's Method: Suppose the autonomous system

$$\begin{aligned} x' &= F(x,y) \\ y' &= G(x,y) \end{aligned} \qquad -\infty < t < \infty.$$

has an isolated critical point at $(0,0)$. If there exists a function $V(x,y)$ that is continuous and with continuous partial derivatives such that .... and ...., then $(0,0)$ is *unstable* critical point.

(b) (1 point) Consider the system the non-linear system of ODE's

$$\begin{aligned} x' &= xy + x \\ y' &= x^{2018} - 2y^3. \end{aligned}$$

Show that $(0,0)$ is a critical point.

(c) (8 points) Determine the stability type (stable/asymptotically stable/unstable) of $(0,0)$ of the system in part (b).

Hint: look for the Liapunov "energy" function in the form $V(x,y) = Ax^k + By^m$.

(try to) HAVE FUN and GOOD LUCK! :)