UPPSALA UNIVERSITET
Matematiska institutionen
Rostyslav Kozhan, Patrik Norén

Prov i matematik
Ordinära differentialekvationer I
1MA032, 2018-04-06

Time: 8.00 – 13.00. Tools allowed: only materials for writing.
Please provide full explanations and calculations in order to get full credit, except for the Problem 1.
The exam consists of **8 problems** *of 10 points each, for a total of* **80 points**. *For grades 3, 4, and 5, one should obtain 36, 50, and 64 points, respectively.*

1. Solve the initial value problem

$$\begin{aligned} &y' - (\cos x)y = e^{\sin x}\sin x, \\ &y(0) = 1. \end{aligned}$$

2. Multiply the equation

$$(3xy + \cos y)y' + y^2 = 0,$$

by a suitably chosen function to make the equation exact. Find the general solution to this equation (which can be in the form $f(x,y) = c$ for some explicit function f).

3. Find the general solution of

$$y'' - 7y' + 6y = 5e^x + 6x - 7.$$

4. Prove that 0 is an ordinary point of the equation

$$(x^2-1)y'' + (x+1)y' - y = 0.$$

Express the solutions of the equation as a linear combination of two power series. On what interval is the solution valid?

5. (a) (5 points) Find the general solution of the system

$$\begin{aligned} x' &= 2x + y \\ y' &= -x + 2y \end{aligned} \qquad -\infty < t < \infty.$$

(b) (2 points) Make a sketch of the phase portrait.

(c) (1 point) Is $(0,0)$ stable/asymptotically stable/unstable as a critical point?

(d) (2 points) Using your result in (b), make a sketch of the phase portrait of the system

$$\begin{aligned} x' &= 2x + y + 2 \\ y' &= -x + 2y - 1 \end{aligned} \qquad -\infty < t < \infty.$$

Continuation on the next page

6. Note: in this problem, leave the coefficients undetermined.

(a) (5 points) *Method of undetermined coefficients* tells us that a particular solution of the non-homogeneous system of ODE's (compare with Problem 5)

$$\begin{aligned} x' &= 2x + y \\ y' &= -x + 2y + te^{2t}\cos t \end{aligned} \qquad -\infty < t < \infty.$$

should be in the following form: $\vec{Y}_1(t) =$ ...... (just an answer is enough; leave the coefficients undetermined; you do not need to compute $\vec{Y}_1$ here).

(b) (5 points) *Method of variation of parameters* tells us that a particular solution of the non-homogeneous system of ODE's (compare with Problem 5)

$$\begin{aligned} x' &= 2x + y + 1/t \\ y' &= -x + 2y \end{aligned}$$

should be in the following form: $\vec{Y}_2(t) =$ ......, where................. satisfy the following system: ....

(just an answer is enough; leave the coefficients undetermined; you do not need to compute $\vec{Y}_2$ here).

7. Let $\alpha \geq 0$ be a positive real parameter. Consider the system

$$\begin{aligned} x' &= -y + x^2 \\ y' &= x + \alpha y + xy \end{aligned} \qquad -\infty < t < \infty.$$

(a) (1 point) Is this a linear system? Is this a locally linear system around the point $(0,0)$? Briefly explain.

(b) (9 points) Classify (by the portrait type and stability type) the point $(0,0)$ as a critical points of this system. Justify your conclusions carefully (Note: you may have to consider separate cases $\alpha = 0$, $0 < \alpha < 2$, etc).

8. Whenever you use Liapunov's method, explain which Liapunov's function you use and which properties it satisfies in order to justify your conclusion.

Let α be a real parameter to be specified later. Consider the system the non-linear system of ODE's

$$\begin{aligned} x' &= y^2 - x \\ y' &= \alpha(xy + y^5) \end{aligned}$$

(a) (5 points) Assuming $\alpha = 1$, determine the stability type (stable/asymptotically stable/unstable) of $(0,0)$ of the system above. Justify your conclusion.

(b) (5 points) Assuming $\alpha = -1$, determine the stability type (stable/asymptotically stable/unstable) of $(0,0)$ of the system above. Justify your conclusion.

(try to) HAVE FUN and GOOD LUCK! :)