UPPSALA UNIVERSITET
Matematiska institutionen
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Prov i matematik
Ordinära differentialekvationer I
1MA032, 2018-08-20

Time: 8.00 – 13.00. Tools allowed: only materials for writing.
Please provide full explanations and calculations in order to get full credit, except for the Problem 1.
The exam consists of **8 problems** *of 10 points each, for a total of* **80 points**. *For grades 3, 4, and 5, one should obtain 36, 50, and 64 points, respectively.*

1. (10 points) Solve the ODE
$$y' = \frac{-2xy - 1}{5y^4 + x^2}$$

2. (a) (3 points) Solve the following ODE: $tz'(t) = -z(t)$.

(b) (7 points) Observe that $y_1(t) = t^2$ is a solution to the equation $t^2y''(t) - 3ty'(t) + 4y(t) = 0$ (on $t > 0$). Use the reduction of order method to find the general solution of this ODE.

3. (10 points) Find the general solution of
$$2y'' - 2y' + 5y = 5\sin 2t.$$

4. Consider the ODE
$$2y'' + (x+1)y' + 4y - x = 0.$$

(a) (6 points) Look for the solution as power series around the point $x_0 = 2$. Find the recurrence relation for the coefficients.

(b) (3 points) Find the first three terms in two linearly independent solutions y_1 and y_2 of this ODE.

(c) (1 point) Justify why these solutions are indeed linearly independent.

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5. Consider the system

$$\begin{aligned} x' &= a_{11}x + a_{12}y \\ y' &= a_{21}x + a_{22}y \end{aligned} \qquad -\infty < t < \infty.$$

(a) (4 points) Suppose $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ has eigenvalue 2 with an eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and eigenvalue 1 with an eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Write the general solution of the ODE system and make a sketch of the phase portrait.

(b) (3 points) Suppose $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ has eigenvalue 2 of algebraic multiplicity 2 and geometric multiplicity 2. Write the general solution of the ODE system and make a sketch of the phase portrait.

(c) (3 points) Suppose $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ has eigenvalue 2 of algebraic multiplicity 2 and geometric multiplicity 1 with $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as an eigenvector. Suppose further that

$$(A - 2I)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Write the general solution of the ODE system and make a sketch of the phase portrait.

6. Consider the system of ODE's

$$\begin{aligned} x'(t) &= \cos y \\ y'(t) &= x \end{aligned} \qquad -\infty < t < \infty.$$

(a) (3 point) Is this a linear system? Is this a locally linear system? Is this an autonomous system? Briefly explain your answers.

(b) (1 point) Find all the critical points of this system (show your computations).

(c) (6 points) Find an equation of the form $H(x, y) = c$ satisfied by the trajectories.

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7. Consider the system of ODE's

$$\begin{aligned} x'(t) &= \cos y \\ y'(t) &= x \end{aligned} \qquad -\infty < t < \infty. \tag{1}$$

(a) (3 points) For the system (1), find the linearization (linear approximation) system at the point $(0, -\frac{\pi}{2})$.

(b) (2 points) Is $(0, -\frac{\pi}{2})$ stable/asymptotically stable/unstable for the system in (a)? Is $(0, -\frac{\pi}{2})$ stable/asymptotically stable/unstable for the system (1)? Briefly justify.

(c) (3 points) For the system (1), find the linearization (linear approximation) system at the point $(0, \frac{\pi}{2})$.

(d) (2 points) Is $(0, \frac{\pi}{2})$ stable/asymptotically stable/unstable for the system in (c)? Is $(0, \frac{\pi}{2})$ stable/asymptotically stable/unstable for the system (1)? Briefly justify.

8. (10 points) Consider the system the non-linear ODE's

$$\begin{aligned} x' &= -x^3 + xy^2 \\ y' &= -2x^2y - y^3 \end{aligned}$$

Determine the stability type (stable/asymptotically stable/unstable) of $(0, 0)$ of the system above. Justify your conclusion.

Hint: look for the Liapunov "energy" function in the form $V(x, y) = Ax^2 + By^2$.

HAVE FUN and GOOD LUCK! :)