UPPSALA UNIVERSITET
Matematiska institutionen
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Prov i matematik
Ordinära differentialekvationer I
1MA032, 2018-10-24

Time: 14.00 – 19.00. Tools allowed: only materials for writing.
Please provide full explanations and calculations in order to get full credit, except for the Problem 1.
The exam consists of **8 problems** *of 16, 12, 12, 12, 12, 12, 12, 12 points, respectively, for a total of* **100 points**. *For grades 3, 4, and 5, one should obtain 45, 63, and 80 points, respectively.*

1. (i) (4 points) Let a and b be two real numbers. Consider the initial value problem

$$\begin{aligned} &x(x-b)y'(x) + (x-3)y(x) = (x-4) \\ &y(2) = a \end{aligned}$$

What does the existence and uniqueness theorem for linear equations say about the interval of existence of the solution of this initial value problem? (your answer may depend on a and/or b)

(ii) (4 points) Convert the ODE

$$y'''(x) - e^{y''(x)y(x)+x} = x$$

to a system of first order differential equations.

(iii) (4 points) Compute e^{tA}, where $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$.

(iv) (4 points) Write the general solution of the Euler equation

$$3x^2y''(x) + xy'(x) - y(x) = 0, \qquad x > 0.$$

2. (i) (3 points) For each of the following equations, indicate (without solving it!) whether the equation is separable; whether it is linear; and whether it is exact (answer only is enough):

(a) $\dfrac{dy}{dx} = \dfrac{x^2 - 2y}{x^3}$

(b) $\left(2y\cos(y^2) - x^2\right) y' = 2xy$

(c) $y' = \dfrac{x}{\sin y + e^y}$

(ii) (9 points) Find the solution of the ODE from (b) that also satisfies the initial condition $y(0) = \sqrt{\pi/2}$.

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3. (i) (4 points) Find the general solution of the ODE

$$xw'(x) + 2(x-1)w(x) = 0$$

(ii) (8 points) Find the general solution of

$$xy''(x) - 2y'(x) + (2-x)y(x) = 0$$

given that $y(x) = e^x$ is a particular solution (Note: ODE from part (i) might be useful in your computations).

4. Find the general solution of the ODE's:

(i) (5 points)

$$y''(t) - 2y'(t) + 2y(t) = 0$$

(ii) (7 points)

$$y''(t) - 2y'(t) + 2y(t) = \frac{e^t}{\cos t} \qquad t \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

5. (i) (2 points) Complete the definition: a point x_0 is called a regular singular point of an ODE $P(x)y'' + Q(x)y' + R(x)y = 0$ if ...

(ii) (10 points) Find a solution of the ODE

$$x^2y''(x) - x^3y'(x) + y(x) = 0$$

in the form of series around $x_0 = 0$ (Note: you do not need to find an explicit expression for the coefficients – just recurrence and initial terms are sufficient).

6. (i) (5 points) Let a be a real number. For part (i) only, assume $a \neq 1$. Find the general solution of the system

$$\begin{aligned} x' &= x \\ y' &= x + ay \end{aligned}$$

(ii) (4 points) Depending on the value of a, classify (by the portrait type and stability type) $(0,0)$ as a critical point of this system.

(iii) (3 points) Make a sketch of the phase portrait if $a = -1$.

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7. Consider the system

$$\begin{aligned} x' &= y - x \\ y' &= x^4 - x \end{aligned}$$

(i) (2 points) Complete the definition: a critical point (x_0, y_0) is called asymptotically stable, if....

(ii) (2 point) Find all the critical points of this non-linear system.

(iii) (2 points) Is the system locally-linear at each of the critical points? Explain.

(iv) (6 points) Classify (by the portrait type and stability type) each of the critical points of this non-linear system. Justify your conclusions carefully.

8. Consider the non-linear system of ODE's

$$\begin{aligned} x' &= 5y^{10} - xe^x \\ y' &= -2xy \end{aligned}$$

(i) (1 point) Find all the equilibrium points of the system.

(ii) (11 points) Use the Liapunov method to determine the stability type (stable/asymptotically stable/unstable) of the point $(0, 0)$.

HAVE FUN and GOOD LUCK! :)