ODE - I

1 MA 032

2018 - 10 - 24 ANSWERS

$\boxed{1}$ (i)

If $b \leq 0$: $(0, +\infty)$

If $0 < b < 2$: $(b, +\infty)$

If $b = 2$: theorem doesn't apply

If $b > 2$: $(0, b)$

(ii) Let
$u_1 = y$
$u_2 = y'$
$u_3 = y''$

Then
$$\begin{cases} u_1' = u_2 \\ u_2' = u_3 \\ u_3' = e^{u_1 u_3 + x} + x \end{cases}$$

(iii)
$$e^{tA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} + \frac{t^2}{2!} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{t^2}{2} + t & t & 1 \end{pmatrix}$$

(iv) Indicial equation :
$$3r(r-1) + r - 1 = 0$$
$$3r^2 - 2r - 1 = 0$$
$$r = \frac{2 \pm \sqrt{4+12}}{6} = \frac{2 \pm 4}{6}$$
$$r_1 = 1, \quad r_2 = -\frac{1}{3}$$

General solution is $c_1 x + c_2 x^{-1/3}$

$\boxed{2}$ (i)

(a) Linear, not separable, not exact

(b) Non-linear, not separable, exact

(c) Non-linear, separable, ~~not exact~~

(ii) $-x^2 y + \sin(y^2) = 1$

3 (i) This is a separable and linear equation
General solution is cx^2e^{-2x}

(ii) Using reduction of order method, we get
$y_2(x) = \left(-\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right)e^x$
so that general solution is $c_1e^x + c_2\left(-\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right)e^x$

4 (i) $Ae^x\cos x + Be^x\sin x$

(ii) $Ae^x\cos x + Be^x\sin x + e^x\cos x \cdot \ln\cos x + xe^x\sin x$

5 (i) ...if it is not ordinary (so either $\frac{Q(x)}{P(x)}$ or/and $\frac{R(x)}{P(x)}$ are not analytic at x_0), and functions

$$\frac{Q(x)}{P(x)}(x-x_0), \quad \frac{R(x)}{P(x)}(x-x_0)^2 \quad \text{are analytic.}$$

(ii) $x_0=0$ is a regular singular point
Indicial equation: $r^2-r+1=0 \Rightarrow r=\frac{1}{2}\pm\frac{\sqrt{3}}{2}$
A Solution: $y(x)=\sum_{n=0}^{\infty} a_n x^{n+r}$ with $r=\frac{1}{2}+\frac{\sqrt{3}}{2}$,

$a_0=1$

$a_1=0$

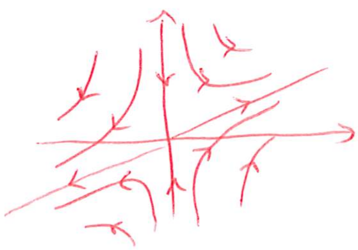
$a_n = a_{n-2}\dfrac{n+r-2}{(n+r)(n+r-1)+1}$ for $n\geq 2$

Note: this is complex-valued. To find a real-valued solution, one should take real part which is a mess...

6 (i) $c_1e^t\begin{bmatrix}1-a\\1\end{bmatrix} + c_2e^{at}\begin{bmatrix}0\\1\end{bmatrix}$ (if $a\neq 1$)

(ii) $a>0, a\neq 1$: proper nodal source; unstable
$a=1$: improper nodal source; unstable
$a=0$: linear source; unstable
$a<0$: saddle point; unstable

(iii) $a=-1$: $c_1e^t\begin{bmatrix}2\\1\end{bmatrix} + c_2e^{-t}\begin{bmatrix}0\\1\end{bmatrix}$



7 (i) ... if (x_0, y_0) is stable and there exists $\varepsilon > 0$ s.t. $\left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| \underset{t \to +\infty}{\longrightarrow} 0$ for any trajectory with $\left\| \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| < \varepsilon$

(ii) $\begin{cases} y - x = 0 \\ x^4 - x = 0 \end{cases} \Rightarrow (0,0)$ and $(1,1)$

(iii) Locally-Linear because $y-x$ and $x^4 - x$ are polynomials, so C^2-functions.

(iv) At $(0,0)$, linearized system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

eigenvalues $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ - spiral sink - stable and asympt. stable

At $(1,1)$, linearized system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$$

eigenvalues $-\frac{1}{2} \pm \frac{\sqrt{13}}{2}$ - saddle point - unstable

8 (i) $(0,0)$

(ii) Use Liapunov function $V(x,y) = Ax^{2k} + By^{2n}$

$\dot{V} = 2kAx^{2k-1}(5y^{10} - xe^x) + 2nBy^{2n-1}(-2xy)$

So choosing $B = 1$, $A = 2$, $k = 1$, $n = 5$, we get V pos. def

$\dot{V} = -2x^2e^x$ neg. semi-def

so $(0,0)$ is stable