UPPSALA UNIVERSITET
Matematiska institutionen
Anna Sakovich

Prov i matematik
KandMa, KandFy, STS
1MA032 ODE I
2020-06-03

Writing time: 08:00 – 13:00. Submission time: not later than 13:20. This is an Open Book Exam which means that you may use the textbook, your notes and/or other resources. The credit for each problem is shown below. For the grades 3, 4 and 5 respectively, one should obtain at least 18, 25 and 32 points (including bonus), respectively. Solutions should be accompanied with explanatory text. Please write your name and page number on each page. The examiner can be reached at 0760480930 *or by email* anna.sakovich@math.uu.se *during the writing time.*

1. (a) (4 points) Solve the initial value problem

$$x^2y' + y = xy, \qquad y(1) = -1.$$

(b) (1 point) On which interval is the solution defined?

2. (a) (2 points) Verify that the equation

$$(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$$

is exact.

(b) (3 points) Find the general solution of the equation in (a). Note that the solution may be defined implicitly, i.e. in the form $F(x, y) = C$.

3. (a) (2 points) Find the general solution of the equation

$$y'' + 9y = 0.$$

(b) (3 points) Find the general solution of the equation

$$y'' + 9y = 2\cos 3x + 3\sin 3x.$$

4. (a) (1 point) Show that the differential equation

$$4xy'' + 2y' - y = 0$$

has a regular singular point at $x = 0$.

(b) (1 point) Determine the indicial equation and the roots of the indicial equation.

(c) (3 points) Find a series solution for $x > 0$ corresponding to the larger root of the indicial equation. Make sure to give the full expression for the series, not just the first few terms.

Please turn the page!

5. (a) (3 points) Find the general solution of the system

$$\mathbf{X}' = \begin{pmatrix} 6 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{X}.$$

(b) (2 points) Sketch the phase portrait of the system.

6. (5 points) Find the general solution of the problem

$$\mathbf{X}'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{X}(t) + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

7. Oscillations in a certain pendulum-like physical system are described by the equation

$$u'' + u' + 2u + u^2 = 0,$$

where $u = u(t)$, $t \in \mathbb{R}$.

(a) (1 point) Rewrite the equation as a first order system.

(b) (1 point) Find all the critical points of the system.

(c) (3 points) Classify the critical points of the system according to their type and stability.

8. (a) (1 point) Which of the following functions is negative semidefinite in a neighborhood of the origin:

$$U(x,y) = x^4 + y^8, \qquad V(x,y) = -15x^3y^5, \qquad W(x,y) = -x^2 - 4xy - 4y^2?$$

Is this function also negative definite? Motivate your answer.

(b) (4 points) Show that $(0,0)$ is an unstable critical point of the system

$$\begin{aligned} x' &= x^9 - y^5 \\ y' &= -x^5 - y^7. \end{aligned}$$

Hint: look for the Liapunov function in the form $V(x,y) = ax^k + by^l$.

GOOD LUCK!